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ASSESSING THE DEVELOPMENT OF GEOMETRICAL THINKING FROM THE VISUAL TOWARDS THE ANALYTIC-DESCRIPTIVE LEVEL

Abstract. The transition to a more advanced stage of geometrical thinking, identified by Van Hiele as the transition from Level 1 to Level 2, is characterised by the gradual primacy of geometrical structures over the gestalt unanalysed visual forms and by the application of the geometrical properties of the shapes. The solution techniques adopted by students of different educational levels and of a range of formal geometrical education experience have been investigated through a number of specially designed for this purpose items. The results indicate that the perceptual strategies are present in the students' strategy choices, including university students. It is suggested that the typical tests that assess the development of geometrical thinking should be complemented with items focusing on this issue.

Résumé. L'évaluation du développement de la pensée géométrique du niveau visuel au niveau analytique – La transition vers un stade plus avancé de la pensée géométrique, identifiée par Van Hiele comme la transition du niveau 1 au niveau 2, se caractérise par une primauté graduelle que les structures géométriques prennent sur les figures perçues dans leur globalité et par l'application de propriétés géométriques des formes. On a mené, grâce à des exercices mis au point dans ce but, une recherche sur les techniques de résolution adoptées par les élèves de différents niveaux scolaires, ayant des vécus variés de l'enseignement formalisé de la géométrie. Il apparaît des choix de stratégies perceptives jusque chez des étudiants à l'université. Une préconisation pour évaluer le développement de la pensée géométrique est de compléter les tests classiques par des questions centrées sur ce point précis.

Key-words. Visual level, Analytic level, Transition, Visual strategies, Geometry I & II.

Introduction

Van Hiele (1986), describing the evolution of his theory since 1955, regrets the fact that initially he "had not seen the importance of visual level", but finds that "nowadays the appreciation of the first level has improved" (p. 41). As Hershkowitz (1990) nicely puts it: "Visualization and visual processes have a very complex role in geometrical processes … More work is needed to understand better the positive and negative contributions of visual processes" (p. 94).

Visual intuition is related strongly to a particular mode of doing geometry: the higher the level of formal geometrical reasoning, the less dominant the visual component becomes. Thus, it would seem quite productive to have also at our disposal the terminology introduced by Houdement and Kuzniak (2003), that

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assigns the strategies of the primary geometrical thinking to Geometry I, the spatial domain of the real world, allowing experimental and perceptual validations and estimations, contrary to the more advanced and rigorous Geometry II, the natural axiomatic geometry modelled according to Euclid's Elements. The use of these last notions of different geometrical paradigms can facilitate the analysis and make more explicit the relation between the geometrical "world" within the learner acts and the particular level of the learner's geometrical thinking development.

The results presented in this paper are part of a wider research attempting to elucidate certain aspects of the aforementioned "positive or negative contributions of visual processes" to the development of geometrical thinking and vice versa (i.e. the effect of geometry learning on the development of visual processes). In this paper we confine ourselves to the results related exclusively to the problem of the transition from Van Hiele's Level 1 to Level 2. Assuming in principle Van Hiele's stage theory combined with the different Geometry paradigms taxonomy, we could phrase (and appropriately rephrase) the question posited in this research as follows:

- To what extent secondary education students have substantially progressed from "visual" Level 1 to the "descriptive-analytic" Level 2?
- Do they *apply* the geometric structures pertaining to Geometry II in a visually differentiated context that reminds them Geometry I? Or more specifically;
- Do secondary students tend to use "visual" (more appropriate to Geometry I) or "analytic" (characterizing Geometry II) strategies to solve tasks which allow both procedures?

1. Theoretical considerations

1.1. Some remarks upon Van Hiele level 2

Certainly, Van Hiele considered as main characteristic of Level 2 the fact that the visual figure recedes to the background and the shape is represented by the totality of its properties. He stressed however that the discovery of these properties should be made by the pupils themselves and not be offered ready-made by the teacher (Van Hiele, 1986). But this is not sufficient: Level 2 is characterized by the pupil's ability to "*apply* operative properties of well-known figures" (*ibid*. p. 41). In a similar vein, Kuzniak & Rauscher (2006) characterise this particular stage as pertaining to Geometry I Prop(erty) towards Geometry II, contrasting it to Geometry I Experimental (where student use measurement and drawing tools to solve a problem) and Geometry I Perceptual (where the solution is based on visual perception). Hence, we should keep in mind that in order for a student's geometrical thinking to be characterised as Level 2, the students should not simply recollect the learned properties of a shape, but should manifest a more active state: a mode of mental activity that strives to both find new properties and *apply* the already known ones to assess spatial relations in a deductive way.

1.2. Type and content of test items

Reviewing the research literature on Van Hiele levels we find that the test items specific to Level 2 can be roughly described as follows: A simple, basic geometric shape (e.g. a rhombus) is shown to the student. Then the student is asked to either "list its properties" (Gutiérrez & Jaime, 1998; Jaime & Gutiérrez, 1994), or identify a particular quadrilateral in a set including a variety of different types (Burger & Shaughnessy, 1986) or select among propositions referring to known properties of basic shapes (Usiskin, 1982).

Considering what have been said above about the importance of student's ability to apply properties, it is clear that this kind of tasks put a rather one-sided weight upon the recollection of properties over their application in novel situations. Hoffer (1983, 1986) and Fuys et al. (1988) considered in their work the skill of applicability of properties. Following Van Hiele, Hoffer (1983, 1986) focuses on the development of "insight" in the students, an ability to perform in a possible unfamiliar situation competently and intentionally. Fuys et al. (1988) revised and extended the initial version of Van Hiele's model. They suggested general descriptors of each level recast in behaviorial terms. These terms were examined and approved by Van Hiele himself. They set additional criteria such as: "discovering of new properties by deduction" (Fuys et al., 1988, p. 65) (Level 3) ("deduction" refers to the ability of geometrical argumentation, without grasping neither the axiomatic structure nor the interrelationships of networks of theorems), "comparing shapes according to relationships among their components" (*ibid*. p. 60), "discovering properties of an unfamiliar class of figures" (*ibid*. p. 60), and "solving problems by using known properties of figures or by insightful approaches" (*ibid*. p. 60)(Level 2).

Another matter of concern is the up to date one-sided dealing with the concept of "congruence" (of line segments or angles) and neglecting other topics, like, for example, the fundamental geometrical concepts of "similarity" or "area", a matter that has also been stressed by Senk (Senk, 1989).

2. Method

2.1. Some general remarks

If we assign a student any geometrical problem in the context of a Geometry lesson, this particular context *imposes* on her/his the need to act in Geometry I or II, regardless of how comfortably s/he feels doing so. The level of the student's competence in some particular topic and her/his demonstrative style of Geometry II have a substantial effect upon the solution the student finally produces or the time s/he consumes for this (and not upon the student's choice of mode of thinking): *all* students will *try* to find a solution anyway pertaining to Geometry II. This research project emphasizes the importance of exactly this issue: the *choice* of the solution strategy. We propose to integrate to the typical Geometry Levels assessment tests the criterion of "*decidability*". In other words, we suggest to consider in the assessment criteria not just the degree of the student's competence to deal with Geometry II structures, but something that acts before that: the student's *preference* to act in Geometry I or Geometry II environment. The student interpretes the visual context of the problem and then decides which solution mode to employ. And this *decision* is not of negligible significance: students should be aware of the unreliability of visual estimates and the superiority of geometrical thought. Hence, it was of considerable importance to structure the problems' context in such a way that it would not impose on the student some particular "strategic choice".

First consequence: the visual aspect of the problem had in any case to counterbalance the geometrical/reasoning aspect of it.

Second consequence: we had to minimize every factor that could lead to any implication alluding to Geometry lesson. The questionnaire was not administered during a Geometry classroom session. The students were told vaguely that a research about "the way students deal with shapes" was in progress, not about geometrical aptitude. The only hint in the instructions we gave: "You can use any method you prefer to answer these tasks. You can use whatever method you deem appropriate, even your geometrical knowledge, if you wish". Finally, we avoided to ask students to justify their answers, in order to prevent the participants to make the conceptual link: "justification"→ "demonstration"→ "Geometry II".

2.1. On the specific tasks

Following the above considerations a number of geometrical tasks were designed. The content of these tasks derived from three essential concepts: (C)ongruence, (S)imilarity and (Ar)ea. Recall that the main idea is to present a spatial problem in visual context different to what can be found in a usual geometry textbook. The correct answer could be found either by some geometrical reasoning pertaining to Geometry II or by a visual estimate. Nevertheless the latter will lead, with the higher degree of certainty attainable, to perceptual misjudgement, due to the inherent limitations of the human's visual system capacities.

This means that the visual setting was constructed in a way that it was as far as possible to produce visual ambiguity or error, rendering the visual estimation precarious. Consequently, the deliberate aim of the setting of the problem was to test the students' *choice* concerning the appropriate strategy (against their presumed knowledge of the limitations of visual estimations), rather than their perceptual efficiency or exactness.

There were two tasks for each topic and two alternative versions (to prevent students from cheating, depending on the classroom conditions), six in all for each student. Fig. 1 (see Annexe) shows the test items related to the congruence of figures (C1 & C2) and line segments (C3 & C4), for the application of the definitional property of circle and rectangle. C2 is a more demanding task, compared to C1, about the congruence of triangles, while C3 is the known Müller-Lyer optical illusion against a background of parallel lines and circle arcs that provide cues for rigorous geometrical reasoning.

Virtually every task concerning congruence and involving visual estimation entails some degree of ambiguity (since visual estimations are reliable on a certain level, see section 4 below). Despite that, we considered necessary to include tasks on this topic, at the expense of complete unambiguity, since congruence is probably the most fundamental concept in secondary education Geometry. Moreover, we thought these test items to be just complementary to the traditional tools and, additionaly, as novel ones, more interesting to be investigated.

Constructing a congruence problem of this kind one faces the following situations:

i) The shapes are congruent and they also seem compellingly congruent. All the typical problems in Geometry II are of this type, providing all the necessary data to prove the asked conclusion without interferring with the visual aspect. Answers based on visual estimates do not count (in Geometry II), as they are irrelevant to the rules imposed by the teacher.

ii) The shapes are congruent but they seem compellingly not congruent. If the problem provides the means to prove that they are congruent, running counter to the perceptual inclination, we have tasks like C1, C3 or C4. In this case the visualizer is "punished" (as far as possible) and the reasoner is "rewarded".

iii) The shapes are not congruent. If the means to *prove* they are *not* congruent are provided, this ceases to be a congruence task and becomes a quite specific and demanding geometrical problem. Thus, it is not possible to construct congruence tasks of this type. This situation can be ameliorated by asking the student, not to prove the incongruence, but at least not to say the congruence (which is compelling to the eye) *for lack of evidence*.

The purpose of C2 is to address exactly this last situation:

(i) The student is provided with the means to prove by geometrical reasoning that triangles A and C are congruent (two equal sides and included angles having sides perpendicular), using the explicit and implicit data of the picture. We could explicitly mention that the points appearing in the figure as circle centers, are really so (the same in C1), but we avoided it, not to overweigh the figure with so many elements. Nevertheless, in case a student asked whether or not the particular points were indeed the centers, as they seemed to be, we immediately provided the information ("yes, they are the centers"). Such instances actually occurred.

(ii) The task asks the student *not* to try to prove triangle B=A or C, due to the lack of sufficient evidence (only one side is equal to a corresponding side of A or C and it is quite apparent from the picture that nothing relates other elements of B to anything of A or C). In case a student tried to prove B=A or B=C, despite the lack of the required data, consuming time in this effort, significantly decreases this student's chance to be included in Geometry II category.

Tasks S1 and S2, and their alternate versions S3 and S4, with exactly the same underlying geometrical idea but different visual context, are shown in Fig. 2 (see Annexe). Tasks Ar1 and Ar2, and their corresponding variations Ar3 and Ar4, were about the concept of area (Fig. 3, Annexe). Note that Ar2 is the same task by means of which Piaget (Piaget & Inhelder, 1960) investigated the relation between length and area.

The paper and pencil test battery included 11 items in all. The remaining five items tested other spatial capacities and were not time consuming: to identify, for example, an asymmetrical figure among symmetric ones or to find the result of a plane rotation.

We allowed 25 minutes for the whole process. Note that, the final choice of the test items, the decision to implement temporal upper bound, as well as other practical details were the result of analysing a number of empirical observations in the preliminary phase of this study. The students involved were from the 1st and 2nd grade of upper secondary school (Lyceum) and the time consumed rarely exceeded 20 min. We added another 5 minutes to ease even more the time limitations. Besides that, we have never been strict on the time limits. There have been cases where one or two said they needed more time to work on some item and we permitted them to do so until the bell rang, even afterwards. The 25 min. time limit is not excessively smaller than the net time remaining if from the official 45 min. of a classroom session we subtract the time needed for the students' entrance and full accomodation, for the introduction of ourselves and our research project, for giving of the instructions, for the collection of papers. Leaving the papers on students' hands after the majority of them had finished, led to attempts for cooperation and influencing each other.

This test questionnaire was administered to 478 students (ages ranging 15 to 23). This sample was composed of two main groups (A and B), each divided in two subgroups (A1, A2 and B1, B2 respectively). For comparison reasons, we sought for a population, inside this age range, that had undergone the minimum possible geometrical instruction, because another research matter was to elucidate the pattern of the effect of various types and contents of the formal geometrical education upon the process of the development of some spatial capacities. A population fitting these criteria was a group (labelled A1) of 80 students attending Bakery-Pastry Practical Apprentiship School (under the supervision of Greek State Organization for the Employment of the Working Force). These students had received some basic geometry instruction in elementary and lower secondary school, but probably their involvement in the educational process was limited. A second subgroup (A2) consisted of the typical tenth graders (154 in number) entering the upper secondary school and ready to attend the Euclidean Geometry syllabus (mandatory for all students of this level in Greece).

In sum, the members of group A were young adolescents with non-systematic instruction in a Geometry II environment. On the other hand, Group B consisted of two subgroups; 150 upper secondary twelfth graders (B1; age 17-18) and 94 Mathematics Department students in Athens University (B2; age 20-23), both subject to substantial and systematic geometrical instruction (adequate knowledge of Geometry II concepts and techiques).

The test battery was administered in the period between April 2004 and October 2005 in the corresponding classrooms.

Since, students were not carrying special geometrical instruments with them when they were given the test questionnaire, all worked in homogeneous conditions concerning this issue. It is true that some students carry with them small straightedges in daily basis, but the comparison measurements required for congruence tasks (C1,2,3,4) could be also accomplished by some offhand, improptu tool: we indeed observed students using the page edge of a copybook or other paper, marking on it the line segments end(s) in order to compare it to another.

We assumed that wrong answers mainly implied either unsuccessful visual estimates or flawed geometrical reasoning. However, in case of a correct answer there was the possibility that a student might have used the visual estimate strategy. This particular difference mattered substantially for the transition to an analytic-descriptive level. As explained above (2.1), a written instruction asking “How you worked it out?" shouldn't be included in the paper test for the following reasons: a) the aim of the study was to test the student's spontaneous reaction and immediate choice without any clue relating the task to any specific Geometry I or II, b) the student's reply "By the eye" doesn't necessarily preclude another more analytic strategy (Geometry II) at her/his disposal as an alternate, second choice, c) it would be of considerable interest to check whether or not this questionnaire could serve as a reliable, convenient and independent instrument for level 2 assessment, and d) for general methodological reasons (for example triangulation; the implementation of more than two methods of observation).

Following these, we interviewed a number of students of the subgroups A1 (14), A2 (101) and B1 (42). The short interview protocol was based on three questions; first question: "How did you obtain the answer to this question?" (for correct answers only); in case the student answered "By the eye", we proceeded to the second question: "Could you imagine a different, more secure, way to solve it?"; in case of a negative answer we asked the third question: "What about using some property of the shapes you see, for instance, this is a circle, etc.".

In order to compare the students' performance in a more typical Van Hiele assessment instrument, we composed two variations of Usiskin's test (1982), each including ten tasks aiming at Levels 1 and 2. These tests were administered to a number of students from groups A2 (70 students) and B1 (50 students).

Finally, for two subgroups (A1 and B1) an indicator of the students' mathematics and geometry formal education competence in school was taken into consideration. For group A1 we had at our disposal each student's marks in mathematics lesson for his/her three years in lower secondary school, of which we calculated the average. The curriculum in lower secondary school includes the above topics of congruence, area and similarity, taught in a transitory way between Geometry I and II. Only a few elements of Geometry II are presented: the basic theorems are only experimentally verified, with the exception of 2-3 quite simple proofs, and the problem solving style is mainly that of Geometry I Prop. For group B1, we considered the average marks in the Geometry lesson for the two years it is taught in upper secondary level (Lyceum).

3. Results

The alternate versions C3, C4 (see Fig. 1), S3, S4 (see Fig.2) and Ar3, Ar4 (see Fig. 3) were administered only to a number of students of subgroups A2 and B1. Differences in performance for these groups across tasks C1, C3 and C4 were insignificant (X^2=1.273, d.f. =2, n.s. and X^2=1.226, d.f. =2, n.s., correspondingly), so we pooled these data, under a general label C1. This was possible also for similarity and area tasks except for the tasks S1 and S3 for subgroup B1 (X^2=11.97232, d.f. =1, p<0.005).

As can be seen from Fig. 1 task C2 demanded more complex reasoning, so we present the corresponding results separately. Group B2 outperformed significantly the other three in task C1 (X^2=41.62, d.f. =3, p<0.0001), S2 (X^2=61.97, d.f. =3, p<0.0001), Ar1 (X^2=43.23, d.f. =3, p<0.0001) and Ar2 (X^2=71, d.f. =3, p<0.0001), all other differences between groups A1, A2 and B1 being insignificant, except in task S2 (X^2=20.54, d.f. =2, p<0.005). The correct response rate (%) for each group and task is presented in Table 1.

Moreover, we calculated each student's total score by counting one point for each correct answer the student produced. ANOVA single factor analysis conducted with these score data showed significant difference between groups (F(4,474)=33.77, p<0.001). Post hoc comparison test between groups limited this difference to B2 relative to the other three and to B1 relative to A1 and A2 (in the A2-B1 comparison using Bonferoni and Scheffé test a significant difference was found (t(474)=-2.68, p<0.0083 and F(1,474)=7.228, p<0.01, correspondingly), but applying Tukey HSD test this difference rendered insignificant (q=3.237)).

<table>
<tr><th>Group / Task</th><th>A1</th><th>A2</th><th colspan="2">B1</th><th>B2</th></tr>
<tr><td>C1 (C3,C4)</td><td>47.5</td><td>58.46</td><td colspan="2">65.53</td><td>90.42</td></tr>
<tr><td>C2</td><td>68.75</td><td>71.79</td><td colspan="2">54.25</td><td>59.57</td></tr>
<tr><td>S1 (S3)</td><td>50</td><td>37.01</td><td>55.31</td><td>16.07</td><td>51.06</td></tr>
<tr><td>S2 (S4)</td><td>1.25</td><td>2.59</td><td colspan="2">14</td><td>32.97</td></tr>
<tr><td>Ar1 (Ar3)</td><td>43.75</td><td>44.80</td><td colspan="2">57.33</td><td>81.91</td></tr>
<tr><td>Ar2 (Ar4)</td><td>21.25</td><td>19.48</td><td colspan="2">30</td><td>68.08</td></tr>
</table>

Table 1: Correct rate (%) of the subgroups in the six tasks.

We present now three examples of the interview procedure (the questions concern right answers only, as already mentioned above).

This student was classified as one having of course an initial response pertaining to a visual strategy, but after probing during the interview, as being able to state geometrical inferences of Geometry II. Despite his initial choice to perform in a Geometry I mode, he is nevertheless able to act in a more analytic level.

Student G.L. (group B1, task C1):

“Interviewer: So how did you work it out?

G. L.: By the eye!

Interviewer: Could you imagine a more secure way to solve it? By the eye you are not so certain, are you?

G. L.: Eh ... eeeh...No ...I think I cannot find something.

Interviewer: Look, how about using some geometrical properties that you know? Here, for example, it says something about circle radii... What do you know about them?

G. L.: They are all equal...So, ...eeh...all these line segments are equal.... hence the rectangles are congruent?

Interviewer: Does it suffice? ...To have two sides equal, I mean?

G. L.: I think so ... Yes.

Interviewer: O.K. You’re right!”

This student was classified as one having the same response before and after the interview. It is clear that she cannot act in a Geometry II mode, and we can assign her to the Geometry I Perceptual.

Student M.L. (group B1, task Ar2):

“Interviewer: And this one how did you work it out?

M. L.: I put the small one [T] inside the big ones [she has on the test paper a T square inside each one of the bigger squares, see Fig. 4, Annexe] and estimated by the eye the remaining area to be equal to that of T…

Interviewer: And in the case of B you found that the remaining area is the same as that of T? Well, this seems quite difficult to me! ... Isn’t there some other more certain and easy way to find it?

M. L.: What else… Nothing comes into my mind.

Interviewer: Maybe something that has to do with the area of certain geometrical shapes?

M. L.: Oh, sir, geometry has never been my strong point!”

The following is an example of a clear case classified as one implementing geometrical reasoning in the initial response.

Student S. K. (group B1, task S1):

“Interviewer: And how did you find the answer to this one?

S. K.: I found the ratio of the two sides and compared these ratios … similar rectangles have the same sides ratio”.

Furthermore, some students had written the corresponding computations on their paper, making apparent the formal geometrical mode of their strategy. See Fig. 5 for some examples of these responses.

One not uncommon misconception has been revealed from the students' responses: taking as line segment's length the number of the dots indicating the units in the segment (Fig. 6). This erroneous mode of measuring combined often with correct geometrical reasoning.

	A1	A2				B1						
Geometrical reasoning in	0 task	0 task	1 task	2 tasks	3 tasks	0 task	1 task	2 tasks	3 tasks	4 tasks	5 tasks	6 tasks
Initial response	100	84.15	12.87	2.97	0	54	24	4	2	2	8	6
During the interview	100	60.39	25.74	10.89	2.97	40	30	12	0	2	6	10

Table 2: Interviews' results (percentages).

The results of the interviews are shown in Table 2. In the first row the number of tasks, in which a student used some form of geometrical reasoning, is indicated. We included in the correct geometrical reasoning even techniques of Geometry I Experimental, such as "measurement by straightedge, compass, pencil, etc.". In the second row the corresponding percentages of the students are presented. For example, 24% of group B1 had an initial response involving some form of geometrical reasoning in only one task or 2.97% of A2 in two tasks or none of A1 in whatever task. After the interviewer's intervention some of the students found eventually another solution and the percentages of the second row have increased. For example, the percentage of group B1 using geometrical reasoning in one task became 30%, while the initial 2.97% of A2 that gave geometric solutions in two tasks became 10.89% during the interview.

According to our version of the typical Van Hiele test and using the "strict criterion" (4 correct answers in 5 questions, see next paragraph), the 42.85% of the sample of group A2 and 61.64% of the sample of group B1 had acquired already Level 2.

The Pearson product moment correlation coefficient between performance in formal mathematical education (specifically geometrical for group B1) and the six tasks of our test found r=0.381 for group A2 and r=0.284 for group B1.

4. Discussion

Attainment of Level 2, in Van Hiele's terminology, seems to be critical for the subsequent progress of a student in more abstract geometrical education (Senk, 1989). Consequently, it is crucial to reliably and validly identify the students' geometrical thinking level at the beginning of upper secondary and university education. In section 1.2 we argued for a broadening both in type and content of the test items employed to identify the students' transition from Geometry I to Geometry II (or, talking from the individual learner's perspective, for Level 2 against the "visual" Level 1).

It is evident that the choice of the criteria and the corresponding tests may have substantial effect upon the assignment of a particular level to a student. In the traditional Van Hiele research paradigm two standard criteria have been put forward: a) the "strict criterion" which means that a particular level is assigned to a student if she/he answers correctly to 4 out of 5 questions pertaining to this level, and b) the "lax criterion", where we have 3 correct answers out of 5 questions.

The choice of the success rate that should be considered as the appropriate qualifier for a student is clearly a matter of discussion. Indicatively, in the current study, we could accept as a "lax criterion" of Level 2 attainment, considering the difficultness of these tasks compared to the traditional instruments as well, 3 correct answers out of 6 (50% success rate). Taking additionally into account the fact that performance in congruence tasks (C1-C4) can be based quite efficiently on visual strategies (as is well established in cognitive science experimental research and actually confirmed by the present results; see Levine (2000) and Newcombe et al. (2000) for an overview), we can set as criterion the following:

Criterion C: "At least 3 correct answers but, in case of 3 correct answers, not 2 of them pertaining both to congruence tasks".

Applying this criterion, only 22% of A2 could classified in level 2, compared to 44% of the traditional test (with application of the "lax criterion"), and only 38% of B1, compared to 62% of the traditional test. These percentages are closer to those revealed by the interviews, where 15% and 40% of groups A2 and B1, respectively, managed to find some form of geometrical reasoning in more than 2 problems. Almost 100% of A1 sample, 60% of A2 and 40% of B1, insisting on visual strategies in all tasks even after probing during the interview, we might say that are still acting in a Geometry I Perceptual mode. In Chart 1 the percentages for all groups concerning our written test are shown.

As clearly mentioned above, setting another criterion would of course result in quite different percentages. Criterion C has been applied post hoc. Between the 14 individuals of A1 group that consented to answer our questions (the rest were not so cooperative), only 2 belonged to the category that passed criterion C and they said (Table 2) that they relied on perceptual estimates. Consequently we cannot know for sure whether between the remaining 10 (passing C) there were some that used different methods (measurements are not improbable).

Furthermore, this particular group had some peculiarities. Officially they ought to be at the same level as A2 (having completed the mandatory 12 years education), but they were not absolutely homogeneous educationally: a few of them had attended one or two years at upper secondary school and then dropped out to seek some vocational instruction.

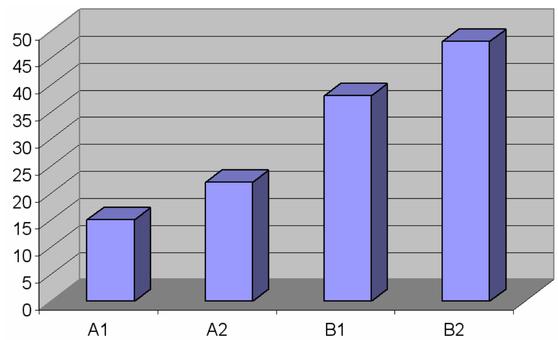


Chart 1: Percentage of students that have attained level 2 according to criterion C (written responses).

Taking into account the idea that different criteria could be applied to different groups we could set a more strict criterion for some students group. For example, a stricter criterion could be:

Criterion C: “At least 4 correct, but only one of C1 or C2”.*

According to this criterion only two members of A1 pass to Geometry II. In any case, considering the interview results, it is not unreasonable to expect that, if we combine these test tasks with some traditional Geometry II items, this percentage will practically be reduced to this low level.

The choice of the component capacities that constitute the so-called Van Hiele Level 2 or the corresponding ability to act sufficiently in Geometry II mode is a matter of interpretation and enrichment of the original theoretical frameworks. Failure to meet “criterion C” may not disqualify completely a student for the Geometry II requirements; we can talk about a “lower degree of Geometry I Prop acquisition" with "strong attachments to Geometry I Perceptual or Geometry I Experimental modes of thinking" and strong tendencies to "regress" any time to them depending on the conditions. Nevertheless the results strongly suggest that for these students the required substructure underlying the more abstract situations of Geometry II, characterised by logical relations and ordering of properties, is not adequately solid. The kind of understanding of higher mathematics they can attain is really a matter of question. This may prove a quite crucial point as far as it concerns syllabus design and the corresponding instructional methods employed.

The results imply that the typical tests putting exclusive stress on recollection of properties of figures or formal definitions (especially Usiskin's test questions pertaining to Level 2, where the questions have not easy quantifiers and the figures do not play any role except in one) rather fail to capture the specific difficulties of the transition from the visual level to a more analytic one. For example, Senk (1989), using Usiskin's questionnaire, found that in her sample consisting of 9^{th} to 11^{th} graders, almost 70% had already mastered level 2. Gutiérrez & Jaime (1998) (using only recognition and classification tasks including recollection of text book properties definitions and simple proofs) reported that the percentage of the 10^{th} graders having attained level 2 and up is greater than 70%.

It is interesting to stress here, in addition to the style of the problem set, the matter of tasks' content: similarity and area were not at all taken into account by the above researchers, while the error rates at our S2 and Ar2 tasks were 85%. See also Chart 2 for a comparison between our test and the variation of the typical Usiskin test.

These comparisons suggest that the percentage of High School final graders acting in a Geometry II mode is lower than until now thought. This is in accordance with Houdement and Kuzniak (2003). Therefore traditional test instruments might be complemented with items like the ones presented here.

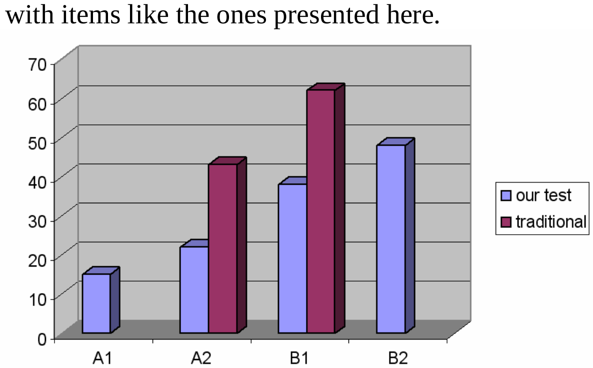


Chart 2: Comparison between traditional Van Hiele test and ours. Traditional tests lead us to take for granted that the majority of High School students have already mastered level 2.

The overall performance of group B2, as was reasonably expected, is higher than that of the others. But the fact that a not negligible percentage of Mathematics students have not yet attained some form of the *geometrical* concept of similarity (see also Kospentaris & Spyrou, 2005) or the relation between square's side and its area, should not pass unnoticed. When the geometrical reasoning demands of the task were harder all groups tended to take recourse to visual estimation. The extensive use of visual-perceptual strategies can be seen clearly by the results in the more demanding task C2 or S1. In these tasks this led to loss of whatever geometrical advantage group B2 might have relative to the other groups, reducing the effect of educational experience.

Visual perception can be quite accurate in estimation of length or distance, as already mentioned above, that fact been established experimentally in cognitive science. The same can be said about the geometrically similar shapes, as this visual mechanism supports the scale invariance of moving objects in the visual field (for a more detailed analysis of the relation of results in S1 task and perception, see Kospentaris & Spyrou, 2005).

Relying solely on visual strategies subgroup A1 outperformed group B in C2 and scored quite above chance level in C1 and S1! (This difference in C2, statistically insignificant as it is, cannot be reasonably attributed to other than chance factor. In the hypothesised case that some B group students tried to find a Geometry II solution and failed because of missing data, time limitations or the difficulty of the required geometrical reasoning, nothing prevented them from giving the last moment a quick visual estimation. We do not have any reason to presume that geometric sophistication diminishes visual abilities (on the contrary). A factor that could explain this difference, if it was not accidental, would be some systematic error in geometrical reasoning of which the solver was not aware, such as that has been observed in S2 (S4) (see below), but no such evidence has been found).

Estimating area, however, is rather difficult, whereas even the "visualizers" in some instances tried to aid themselves by some kind of pictorial computation (for example, the case of student M.L. above was not unique).

It is noteworthy at this point to mention that the above given examples of geometrical misconceptions about a side's length (Fig. 6, Annexe) were not the only ones observed. Another common misconception was that of misidentifying "similar shapes" as those "having the same area" or the linear (proportional) relation between shape's side and area. These and other, like, for instance, a more idiosyncratic use of corrupt geometrical "knowledge" or effort to apply some absolutely irrelevant geometrical tool (Fig.7, Annexe), had of course a negative contribution. In other words, in order for geometrical reasoning to be productive it has to be correct; otherwise it is worse than naïve visual thinking. These specific misconceptions are more likely to be revealed in such tasks. Consequently, we argue that they should be taken into account in educational design.

Returning to the difficulties observed in B2, it would be relevant to refer to Van Hiele's (1986) view that a person after having attained Level 2 even in *visual thinking* the formed structures of this level are always at his/her disposal, but with one exception: if he/she thinks in another context. But this effect of visual set seems not to be evenly distributed among tasks and may be related to the content of the task (so it can be attributed, for example, to specifically inadequate or ineffective instruction in similarity and area). This raises the following question: Have these students really attained some adequate capacity to perform in Geometry II mode, but have trouble applying simple structures of it in a visually differentiated context? Or are they still acting as in the initial stages of Geometry I (Perceptual and Experimental), at least for these particular topics? Previous research (Mayberry, 1983; Gutiérrez & Jaime, 1987) in fact revealed that preservice elementary school teachers usually act in Van Hiele Level 1 or 2, but B2 is a student's group derived from a mathematically sophisticated population.

Another result worth noticing is the marginal difference between B1 and A2, despite the great amount of geometrical instruction the former have received (3 or 2 hours per week for 2 years in Euclidean Geometry and 5 in Analytic Geometry during 11^{th} grade for almost 80% of them) and the observed low correlation to school geometry performance.

Van Hiele himself appears to have retained his initial view that "in practice thanks to education nearly *all* pupils attain the two levels [first and second] sooner or later" (1986, p.44). Interestingly, Piaget & Inhelder (1960) claim that reaching Stage IV children understand the appropriate relation between a square's side and its area and can give an adequately approximate answer to this task (Ar2). Piaget (*ibid*), studying the formulation of the concept of geometrical similarity and its relationship to the concept of proportion, accepts that Stage IV is marked by the attainment of true proportionality. Hence, according to Piaget, at this particular age range (15-25), and since Stage IV is attained by almost *all* children around the age of 13, we sould expect for S2 and Ar2 tasks the overwhelming majority of students to find a solution more oriented to Geometry I or II. For Piaget the passage to higher levels of spatial cognition is somehow inscribed into the order of things and constitutes an ineluctable part of the developnmental process (we could probably characterize his view as a more nativistic one). On the other hand, according to other schools of developmental theorists (like, for instance, Van Hiele) the cultural enviromental context (in our case, schooling) acquires a far greater importance.

The results of this study strongly support Van Hiele's thesis about the irreplaceably crucial role of school instruction upon the formation of the fundamental geometrical concepts (see A1 results). Transition from Level 1 to Level 2 is an arduous process demanding great instructional effort, without which students remain forever at level 1 (group A1). But as far as concerns progress to level 3, that presupposes a complete mastering of level 2, or the percentage of the students' population that substantially reach this stage, things do not allow us to take anything for granted. Maybe we could attempt to put forward some tentative explanation for this obvious discrepancy between former and recent studies: it is not unlikely that pupils grown in the particular educational and cultural milieu of pre-war and first after-war decades in European countries had developed a cognitive style more familiar to formal, abstract geometry. But nowadays children are more tuned to the visual, holistic style of TV watching and computer gaming, and this influence has considerably reduced the analytic parameter of their thought (see also Healy, 1999).

Considering that the main goal of secondary geometrical education is to at least provide the students with an adequate knowledge of Geometry II structures, a necessary prerequisite in order to enter into the Geometry III paradigm, do we have before us an evident educational inefficiency? These tasks and students' responses show, in the best of cases, that geometrical concepts and procedures of Geometry II are not "getting out" of the typical textbook context to "function" in real world situations (Geometry I). It is likely that these structures are quite precarious and easy to disintegrate with the context change.

A plausible description of the processes taking place is this: the majority of the students finishing elementary and lower secondary education remain at Level 1 or, in the best case, at a low degree of Level 2 acquisition. At the beginning of the upper secondary school the syllabus and the instructional methods put them suddenly in the Geometry II "world". Most students find some way to cope with the situation, mainly by replicating the algorithmic procedures presented by the teacher without real understanding, but the connection with the concrete spatial situations of Geometry I is lost. The structures of the lower levels remain unaffected. The observed above low correlation with the school marks in formal Geometry and a previous finding of geometry levels research that some students appear to act at two levels simultaneously, tend to support this view. Students appear to replicate some of the superficial aspects of Geometry II paradigm, whereas their geometrical thinking remains deeply into Geometry I.

Working in Geometry II from an early stage is probably the source of these problems, as Van Hiele warns (1986, p. 66):

> It is very usual, though always condemnable, to speak to pupils about concepts belonging to a level that have not at all attained. This is the most important cause of bad results in the education of mathematics. The result of such instruction is that the pupils are obliged to imitate the action structure of the teacher. By doing so they usually succeed in mastering operations belonging to the level. But because the

> action structure does not result from a real understanding (i.e. not by analysis of lower structures), it must result from a global structure of acting. The success is seemingly complete: In the long run the pupil is able to calculate as fast as the teacher.
>
> The teacher does not use low (visual) structures when he computes, neither do his pupils. But whereas the teacher (as we suppose and hope) has obtained the knowledge of computing by a transformation of the structures of a lower level, with the pupils such a relation is absolutely absent. With the teacher, computing will generally be connected with concrete material. He will, in a new concrete situation, usually be able to apply his knowledge. In such cases, however, the pupils be powerless.

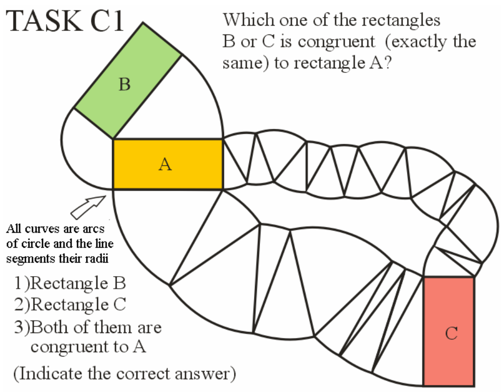
Besides the need to extend the period dedicated to Geometry I, another crucial point is the instructional method: situations, activities and adequate time that give the opportunity to the student to discover the relations and structures of this domain by himself, seem indispensable. Van Hiele stresses this point also (*ibid*. p. 63):

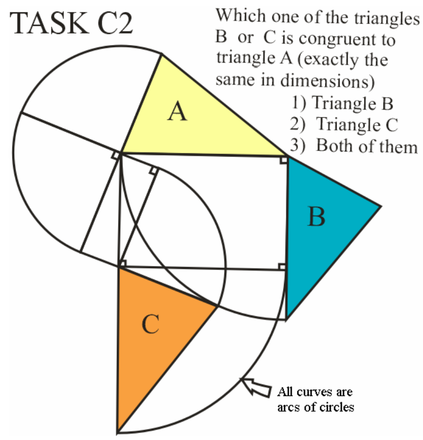
> If pupils do not find the network of relations of a given level by themselves, when starting from a concrete situation, will have difficulties returning to the corresponding signification in the developed network of relations, unless the concrete situation happens to be that of the teacher's original situation.

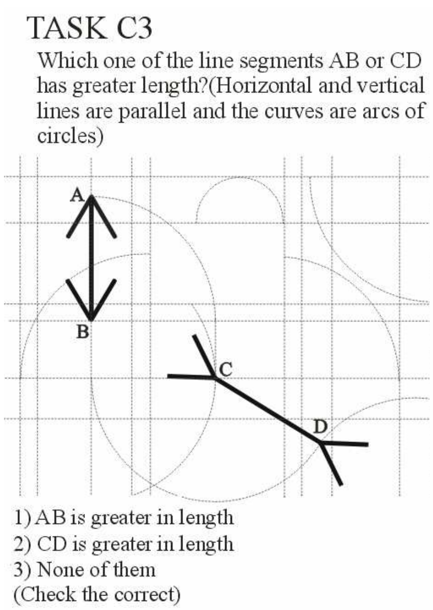
Insistence to present ready made the deductive structure and logical relations of Geometry II immediately at the beginning of upper secondary school without reliably assessing students' level, as things continue to happen until today in Greece, implies that we take for granted that they mastered level 2. The above results posit serious questions about this assumption.

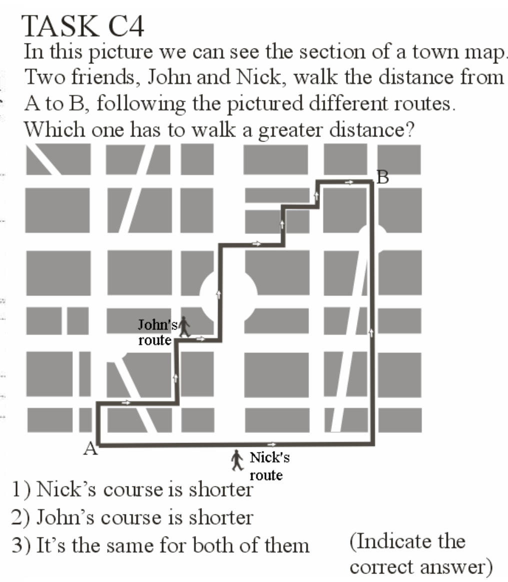
Annexe

TASK C: Congruence
(Pictures are scaled-down to approximately one half of the original).

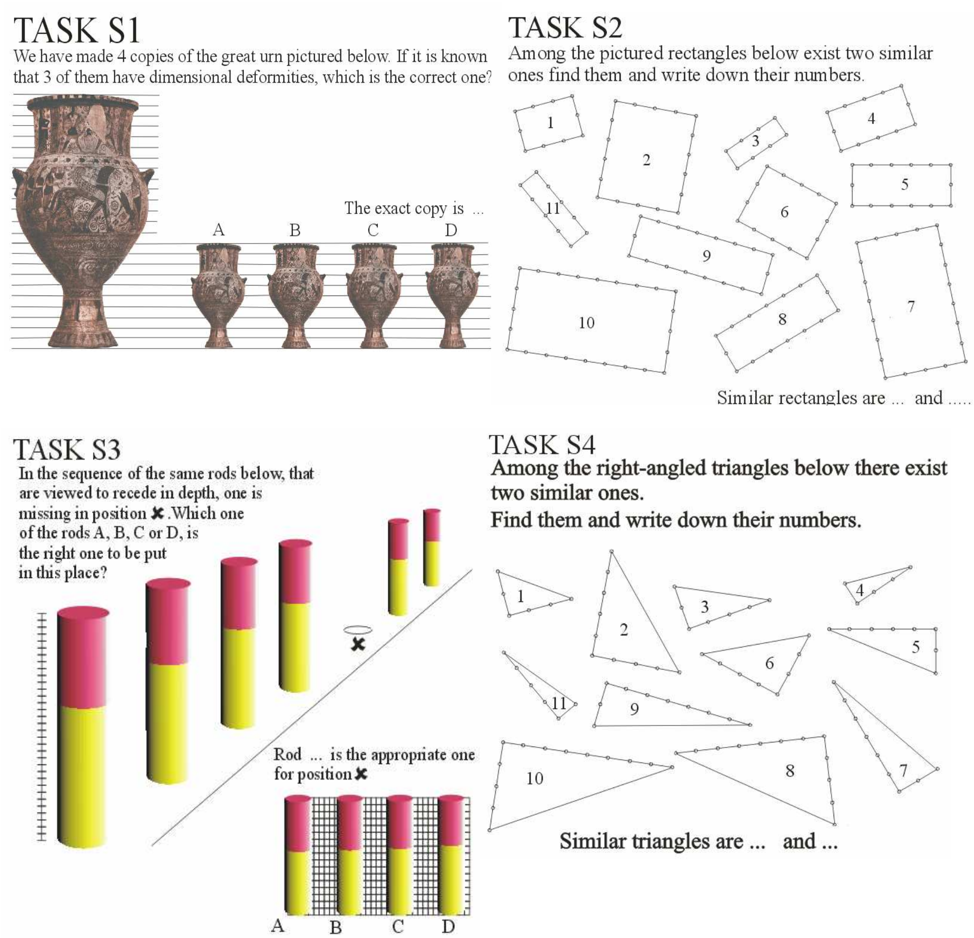






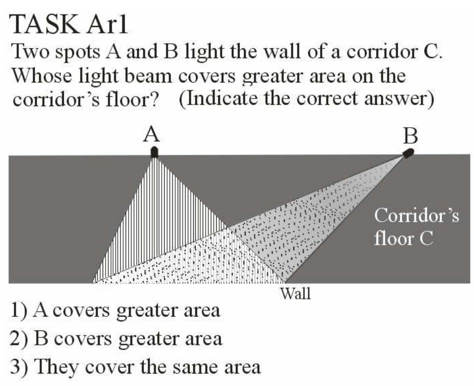


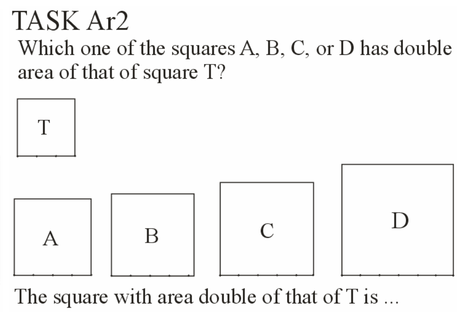
TASK S: Similarity
(Pictures are scaled-down to approximately one half of the original).

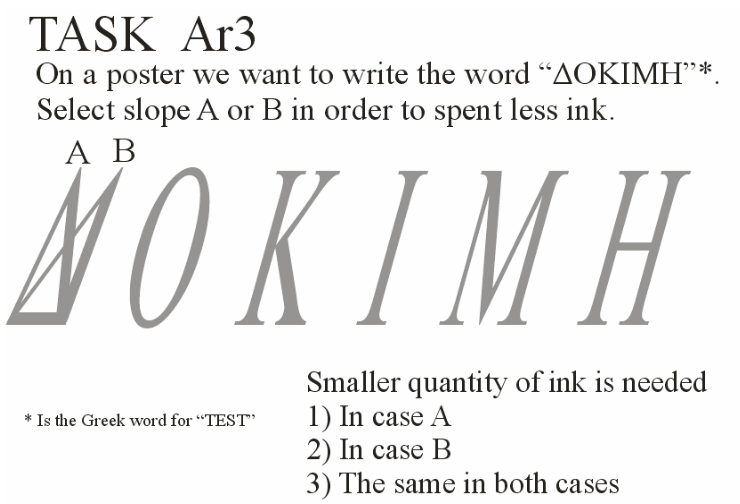


TASK Ar: Area

(Pictures are scaled-down to approximately one half of the original).







TASK Ar4

Exactly the same as Ar2, the sole difference being the length of the square's side (5 units)

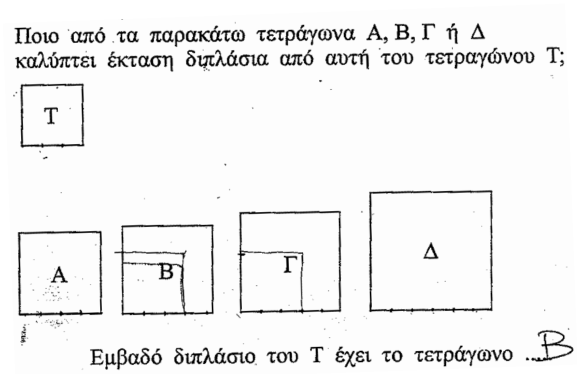


Fig. 4: Student's M.L. (group B1) response, a case of visual estimation (with correct results!).

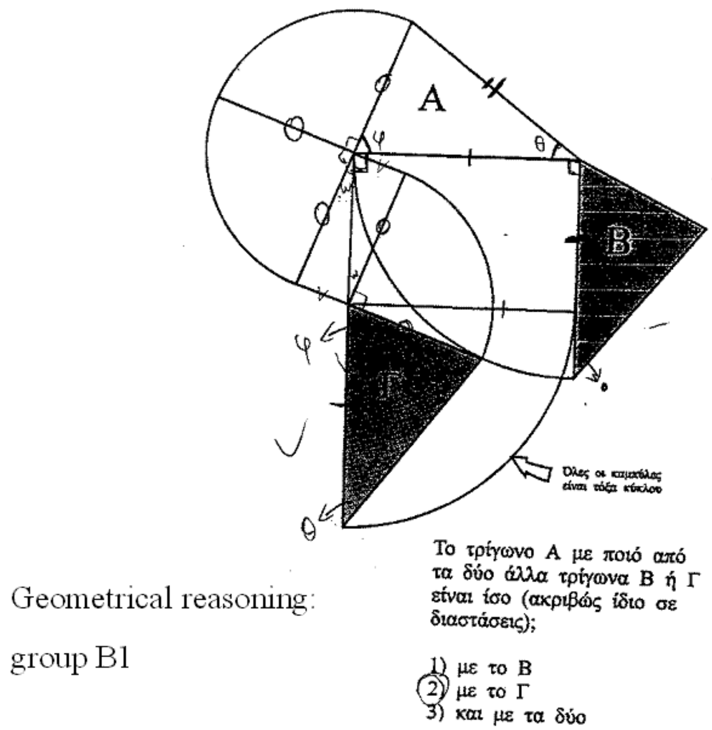


Figure 5a

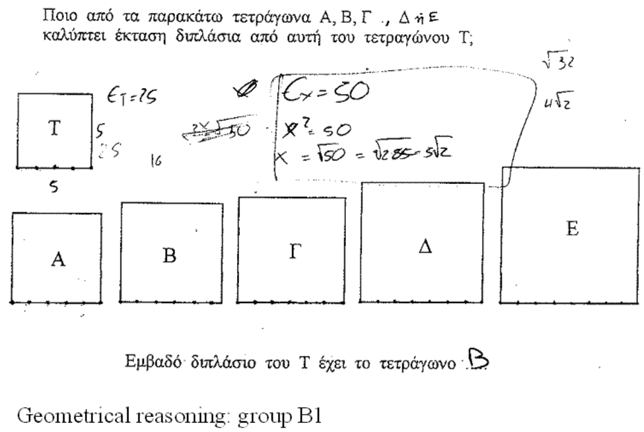


Figure 5b

Figure 5: Students' responses with explicit geometrical computations.

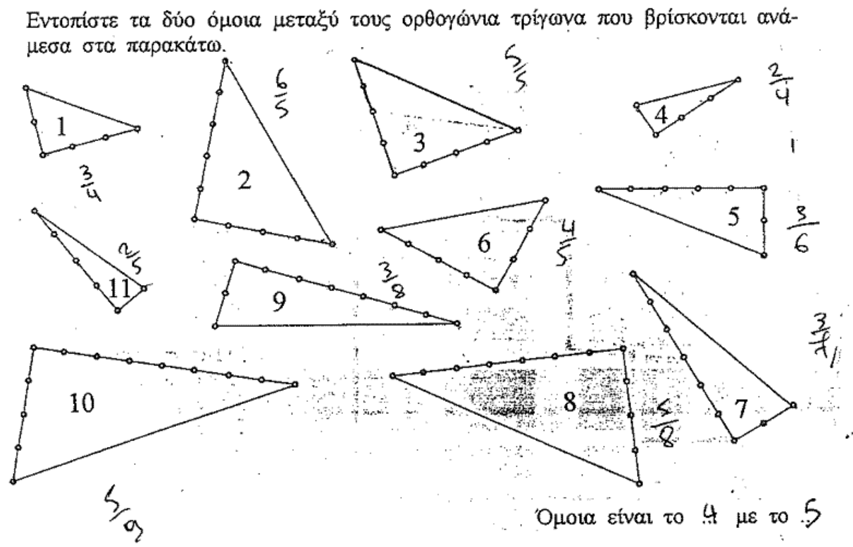


Figure 6: A case of correct geometrical reasoning with erroneous measurement.

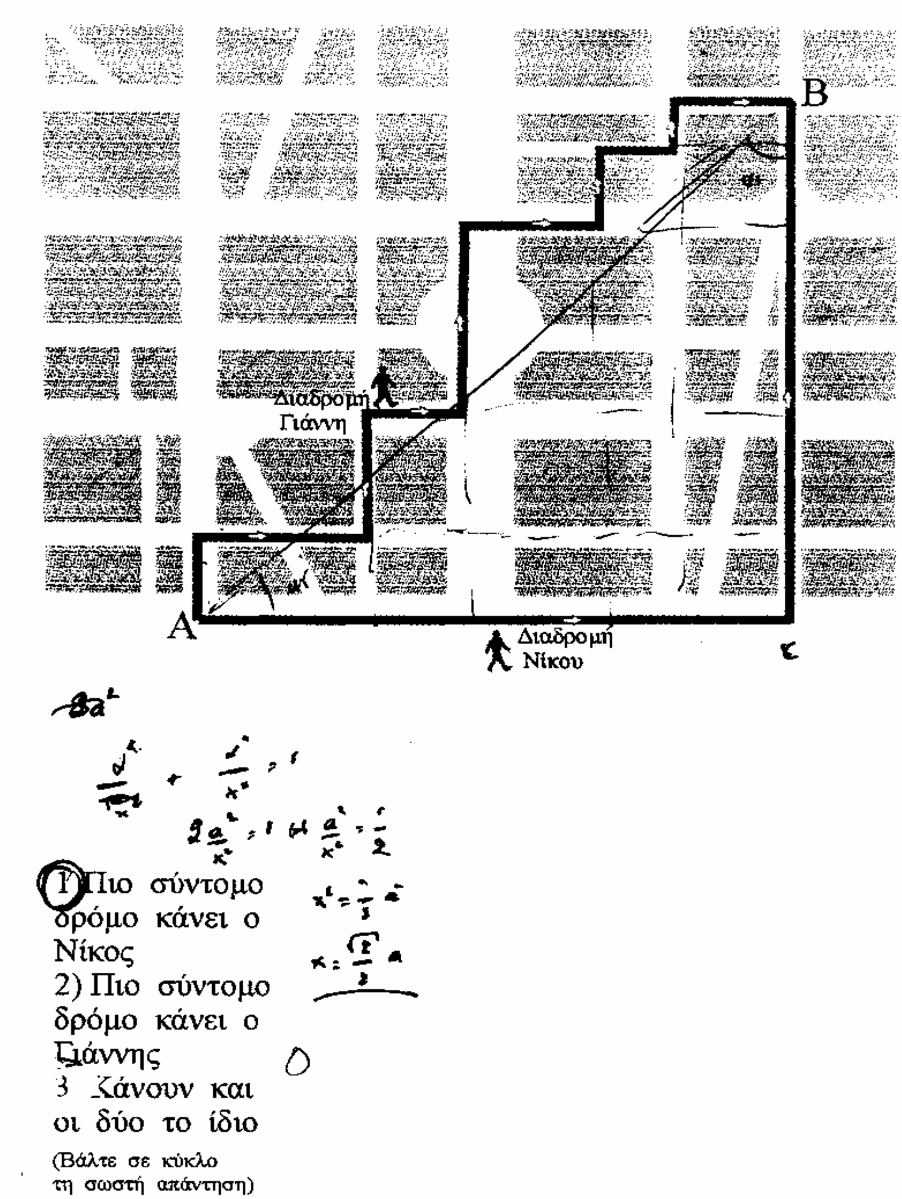


Figure 7: Indiscriminate application of unrelated geometrical reasoning (group B1).

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