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A Reading of Euclid's Elements as Embodied Mathematics and its Educational Implications

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ABSTACT: We suggest that "embodied mathematics", should be studied in an historical context as well. Some early mathematical results that have later evolved to the first theorems in Euclid's Elements have arisen as attempts to render logically some main categories of our perceptual system. We propose an alternative reading of main themes of Euclidean Geometry and suggest some considerations giving a bodily character to concepts in primary teaching of Geometry.

Key words: Archetypal results, conceptual system, embodied mathematics, objectification, pattern, perceptual system, prototypes.

INTRODUCTION

An important question about the nature of geometrical thought, having immediate influence on teaching, is whether it is "transcendental" and works in a world of platonic objects or does it include a wide field of compressed bodily actions.

The theory of embodied mathematics was proposed recently by Núñez & al. (1999) and Lakoff & Núñez (2000). The relevance of human body to our conceptual systems is decisive in the theory of embodied mind in general, as a theory in which the correlation between human experience and cognitive sciences is attempted, Varela & al. (1992). Lakoff (1987, p. 364) started to discuss embodied mathematics very early, as it appears when he states: "mathematics is based on structures within the human conceptual system, structures that people used to comprehend ordinary experience". On the other hand, embodied mathematics allows us to address the main epistemological problem of mathematics:

Human mathematics is embodied, it is grounded in bodily experience in the world… not purely subjective… not a matter of mere social agreement…
It uses the very limited and constrained resources of human biology and is shaped by the nature of our brains, our bodies, our conceptual systems, and the concerns of human societies and cultures.

(Lakoff & Núñez, pp. 348 –365).

The embodied mathematics has been supported through a sophisticated theory by Tall (2002, 2005) in a wider context, attempting a classification of mathematical thinking through the use of an interesting composition of neo piagetian theories. In his work we come across a fundamental model of development of Mathematics connecting perception to conception. In the case of Geometry, a concrete theory is introduced that joins the actions on the environment and the perception of the world with the "perfect" or platonic mental objects:

The *embodied world* of perception and action, including reflection on perception and action, … develops into a more sophisticated Platonic framework.

Watson-Spyrou-Tall (2000)

Following the above current trends, we suggest a "descriptive model" of understanding these reductive theories, in particular embodied mathematics towards the formations of geometrical concepts in the prehistory of Geometry and in Euclid's Elements (especially in Book I). This point of view seems to be very little discussed and developed so far.

We assert that the main mechanisms such as the perception of gravity and the recognition of shapes provide the context for the construction of the central geometrical concepts. The idea that the basic factors for the perception of space are the *vertical* (briefly V), the *horizontal* (briefly H) and the recognition of visual shapes (briefly S), seems to belong to Piaget (1956) who also connects S with the conceptualization of proportion. We attempt to show how the main theoretical instruments of Euclidean Geometry are produced around *verticality, horizontality* and *geometric similarity* (briefly VHS). According to this view Pythagorean Theorem and Thales' Theorem about similarity are the conceptual representation of our sense capabilities.

THEORETICAL TOOLS

Prototypes in Geometry

In order to describe the embodied situation, we have to reflect on perception, not only in the context of contemporary neuroscience but also in the light of historical sources. This *genetic* approach inquires into potential environments, the necessary metaphors and ideas that have preceded the scientific formation in mathematics, as we know it today.

The idea of encapsulation of actions on environment in the constructive process of concepts was supported initially by Piaget. In addition, Lakoff stresses the linguistic function of metaphors that intervene in our conceptual systems. Thus the mind is understandable as a "metaphor machine", Lakoff (1987). In this respect, a linguistic term has been introduced; that of a *prototype* (at first by Rosch, 1978). A *prototype* is regarded as the 'best exemplar' of a concept, as a specification for the members of a family and in this sense; it is the ideal core of a concept, Harley (1995, p. 193), Malt[1] (1999, p. 333).

The views of Tall & al (2000) are specified for the geometrical thinking (including the prototypes) and could be summarised in the next schema:

Perception of the world includes the study of *space and shape,* which eventually leads to *geometry,* where verbal formulations support a shift to Euclidean proof: Interactions with Environment → Perception → precepts real- world prototypes → platonic objects & Euclidean proof.

The psychological ground of the development of Geometry has been described:

> This is rooted in *perceptions* of objects in the world, initially recognised as whole gestalts. Some are specific individuals perceptions, … but more often are perceived as *prototypes* that apply to a wide range of precepts…
>
> (ibid, p. 83)

Space Perception

For the understanding of the primitive geometric concepts as embodiment, another web of ideas is needed which distinguishes the main components of our *perceptual system*. Subsequently, we will focus on three functions that constitute an important part of our perceptual apparatus, i.e., the sense of the VHS. Altogether they are connected functionally, affect decisively our adaptation in the environment and offer us the main mental tools to describe our experiences of the world.

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[1] *"The idea that there is some* core *part of meaning that is invariant across all contexts or instances of a category offers a useful solution to this problem in principle, but in practice, cores for many words may be difficult or impossible to identify, just as were defining features…For instance, that the meaning of the word* line *is subtly different in each of many different contexts (e.g., 'tanding in line', 'crossing the line', 'typing a line of text' and that the variants are constructed at the time of hearing/reading the word from some core meaning of the word in the combination with the context in which it occurs", Malt (1999, p. 333).*

The sense of V is characteristic of our substance as intelligent beings and is inherent in our *perceptual* bodily experience in a nature dominated by gravity[2]. Note that Merleu–Ponty, as well as Piaget, have underlined the significance of gravity in the comprehension of space and their position has been and ever since supported by others: Zuzne (1970), Ibbotson & Bryant (1976), Varela & al (1992). According to Lakoff (1987, p. 277) verticality is the main source domain and is connected with our understanding of quantity as well. For the apprehension of the vertical an affinity to the horizontal has been suggested in psychology: "the acquisition of the vertical is synchronous with that of the horizontal", Piaget & Inhelder (1956, p. 400), Mackay & al. (1972).

Similarity is, generally, a far broader perceptual category: it appears either as visual (for figures or colouring) or auditory etc., and consists of a general trend both in the perceptual and conceptual systems that unifies the manifold of experience into rules. The recognition of visual shape tends to be a particular function of the skill of apprehending similarity. In fact, Gentner & Medina (1998) consider "similarity" as a classification mechanism: "a *process* of comparative reasoning and the similarity as a *product* e.g. a sense of closeness or representational unity". It is important to notice that since similarity appears as a quality function belonging to subjective impressions and internal representations, it is difficult to communicate.

Conception via Objectification: Archetypal Results

In our presentation, we need a gross meaning for the "objectivity" of the mathematical concepts, as it is used by mathematicians: one with respect to their ideality (as platonic objects), and another as social agreements tested in the context of human experience. In both cases we could say that mathematical concepts in any civilization stand for their inter-subjective knowledge and expressions, Netz (1999).

We think that the first results of Geometry arose from the persistent effort for an objective rendering and reification of functions connected with VHS forms, via arithmetic and logical relationships next. These functions constitute properties of our spatial perception and we can represent using figures. Besides, in order to do mathematics, we need at least two representational systems and the translation to each other (Duval, 1995). Mathematics is closely tied to the logico-mathematical ability of counting. Through numbers, experience becomes homogeneous, inter-subjective and easier to transmit. Number stands for a certain grasping of the world's truth. Humankind's apprehension of the idea of number was a first instrument to obtain "objectivity". Thus, such results as the Pythagorean triads and the internal ratio of the legs in right triangles, in which the "objectivity" is given by an arithmetical relation,

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[2] *"the phenomenal orientation of the form is determined by directions in environment. These directions are supplied by the pull of gravity, the visual frame of reference, or instructions", Zuzne (1970).*

appear as "archetypal" of the later basic theorems of Geometry. Such "archetypal results" are the common origin of oriental mathematics, since almost similar problems had been treated by Egyptian, Babylonian, Chinese or India's civilizations.

TOWARDS THE FORMATION OF GEOMETRICAL NOTIONS- AN EMBODIMENT READING OF EUCLID'S ELEMENTS

The point, the straight line and the plane

In the sequel we use the theory of prototypes in a specific way. We preserve a new hierarchy, from the "elementary" prototypes that depend only on verbal description, to the more complicated that, furthermore need a connotative structure in order to be understandable.

Our proposal is that *points, lines,* and *planes* in their ideal use, serve as our main examples for "elementary" *prototypes* in Geometry, (in the sequel simply *prototypes*). We investigate these concepts as being formed by the influence of linguistic metaphors. According to Piaget, straight line appears to be nothing more than an encapsulation of the traveling. Lakoff & Núñez (2000) give some hints for the notions of point, line and plane and connect them with the Basic Metaphor of Infinity (BMI), suggesting an intelligent construction.

We make here some remarks arising from the history of mathematics. In Euclid *point* (:το σημείο) and *straight line* (:ευθεία γραμμή) as well, are included in *Terms,* (:Όροι). The used terminology asks for our attention as a linguistic metaphor since it composes something wider than simple definitions where they have the meaning of an access of a concept in a category[3].

The point in Euclid arises as an *exemplar,* in different forms. The extremities of a line are points, the intersection of two lines is a point, etc. Zervos (1972, p. 431). The point is an invention that holds the discussion and the description of a problem in space. Such an expression as "Let a point" is a suggestion to be accepted and thematised as a mental object in a discussion.

In order to enforce the arguments of this paragraph we refer to Tall's hint:

[3] Indeed the *Όροι* in Greek are called " landmarks or boundaries, but inscriptions on obligations of an economic nature following some agreement", as well. Besides, the Greek word for "*to define"* (:*ορίζεσθαι,* provided from *όρος*), also means *to mark off*, the Form or (*Είδος*) of an object from that which *was not* Szabo, (1978, p. 256). The *σημείο* – has the meaning of *sign,* as well and is given "*A point … it can be grasped by understanding only",* Russo (1998). Furthermore, we should add a linguistic product in Greek, the verb σημειώνω (I note), but the current term semiology as well (Duval, p. 2).

Language plays an increasingly subtle part in this geometric development. Prototypical shapes such as a straight line, a triangle, a circle, are described verbally in ways that support the imagination of prefect platonic representations, such as perfect straight line with no width that may extended arbitrarily in either direction, or perfect square, a perfect circle. Thus, paradoxically, perfect geometric entities depend on language to construct their meaning.

D. Tall et al (2000)

Angle: a compound Prototype

The notion of an '*angle'* is crucially involved in the fundamental perceptual categories, as they are presented in Section 1. 2, inasmuch as it obviously mediates in shape's recognition. In almost every early foundational process, a prominent role for the *angle* is devoted by mathematicians and thus it turns out to be an unavoidable term in any kind of geometric reasoning. In Euclid's Elements (I, Definition 8 and 9), the angle is defined as follows:

A plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.
And when the lines containing the angle are straight, the angle is called rectilinear.

(quoted in Heath, 1956, p. 176)

Therefore at least within the scientific status of the theory, angle is not given as a measurable magnitude. It rather occurs as a *compound prototype*, given by verbal descriptions and refers to the simpler basic prototypes of points, straight lines and planes plus Euclid's invention of the linguistic term *inclination*. The compound prototypes are anticipated by the theory of prototypes[4].

Angles are only amenable to the single manipulation of superposition and coincidence, in the same uniform way that occurs *for all figures* in Euclid's Geometry. These synthetic conditions do not constitute objective criteria for angles, independent of subjective experience, like those provided by measurement. Besides, two angles are similar if and only if they are congruent. Thus angle becomes a tool that intrinsically participates in the determination of the shape [5].

Note that in post-Hilbertian considerations, an acceptable measurement theory for angles is not at all obvious and turns out to be a deep result in mathematics and, also,

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[4] A definition is not a matter of giving some fixed set of necessary and sufficient conditions for the application of a concept; instead, concepts are defined by prototypes and by types of relations of prototypes. Rather than being rigidly defined, concepts arising from our experience are open – ended, (Lakoff & Johnson, p. 125).

[5] The use of the term "similar" for angles is common (according to Proclos) in Thales (Heath, 1981, Ch. 4, 4.b) and is alive even in the definitions of solid geometry (Euclid's Elements XI, definition 10).

difficult enough to be explained in neurological terms[6]. However, through a sophisticated treatment in Euclid's Elements, angles are subject to enactive manipulation, without any mediation of measurement or theory:

> In fact Euclid does not use any relation which is not reducible to coincidability between lines until he treats ratios.
>
> (Mueller 1981, p. 41)

In the case of perpendicular and parallel lines, either the definitions or their verification are reduced to certifications concerning prototypical configurations, such as points, straight lines and angles.

In particular: "a straight line meets another straight line perpendicularly, when the formed angles are equal – then the angles are called right angles" – "All right angles are equal", (Elements I, Definition 10 and Postulate 4). In the above statements a twofold purpose is achieved; On the one hand we have an identification condition for perpendicularity, via right angles (see Figure 1); on the other hand, through the use of 'all' in the above assertion, an implicit transcendental definition is formulated.

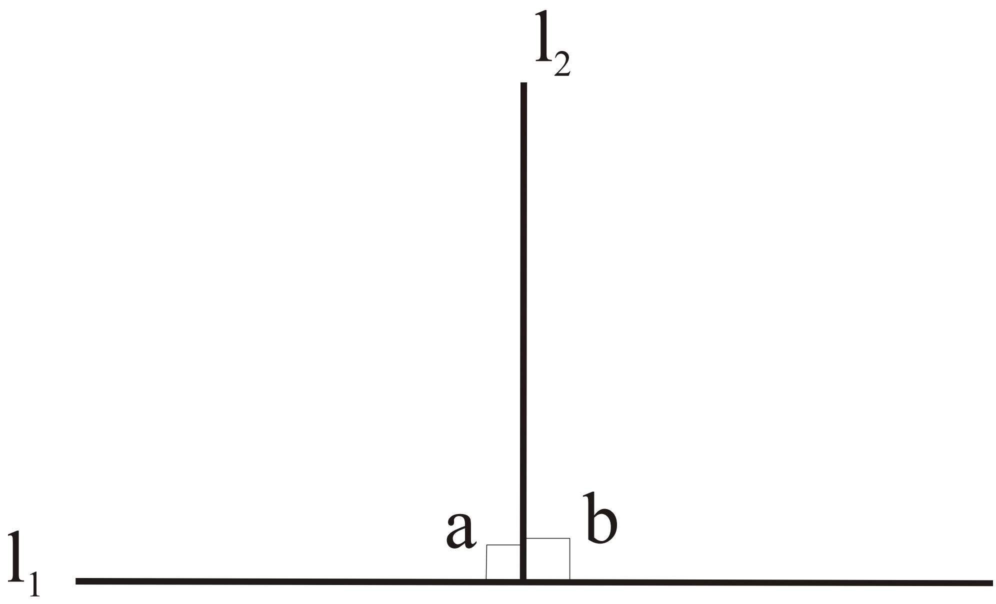

Figure 1: Lines l_1 and l_2 are perpendicular iff "a = b" (1∟)

[6] They are special cells in the perception of angles. The recognition of the angles is a sort of innate cognitive apparatus, as evidence see in R. N. Haber & M. Hershenson (1974, p. 358) "…cells have been found which respond to the angles between two lines, rather than to the lines alone", (p. 55) and about the infant's perception of angles. See also in (Wenderoth P. & D. White, 1979).

In the case of parallel lines, their definition includes a transcendental term, such as ‘produced indefinitely’ (Heath, 1956, p. 190). We remark again that the formulation of the 5th Postulate (ibid. p. 202) permits a finitistic type of argument, referring again to straight lines (see Figure 2).

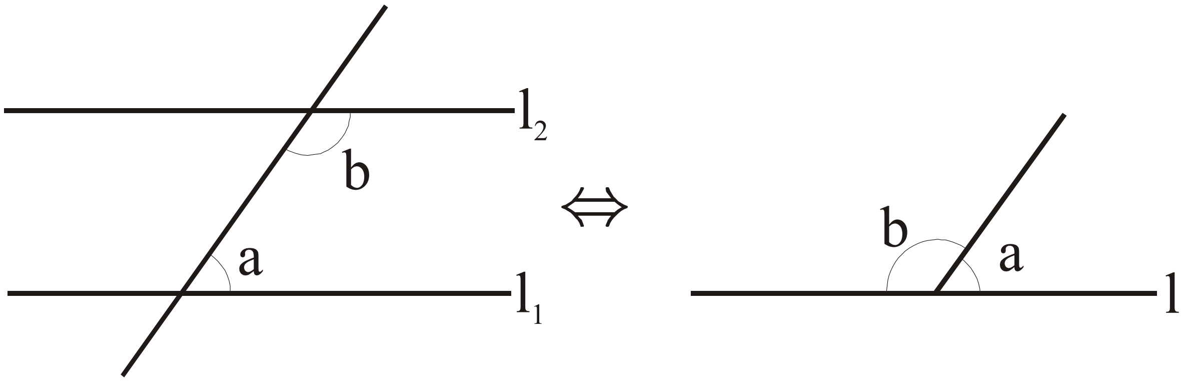

Figure 2: Lines l_1 and l_2 are parallel iff “a + b” would form two right angles

We consider all the above as indicative and exemplary cases of reduction to prototypes. In fact, it is enough to verify that after a suitable ‘displacement’, we arrive at an arrangement that proves to be a straight line.

Geometric Patterns: The Triangle

We suggest that between the primary notions articulated via elementary prototypes and the archetypal results an additional conceptual structure should mediate. This process is carried out by means of a constructed hierarchy referring to geometric notions of growing complexity, for which we reserve the term *pattern.* With respect to Geometry, the pattern of a *triangle* appears*,* as it involves a minimum of data from the system of prototypes (as they are point, line, plane and angle) in a unified way, as a coherent structure. In the study of Solid Geometry, next to this classification could be the tetrahedron, a distinct spatial pattern.

As a pattern, the triangle is the first step for the analysis of a figure and its reduction to prototypes. In a triangle, the broader geometric relations of equality, similarity and area, bestow a quality of additional structure and bring forth more associations in plane geometry. Thus, the pattern of a triangle is fundamental and it becomes the principal instrument, mediating in all proofs concerning even more complicated configurations. According to this point of view, the pattern of the triangle is a necessary and decisive factor for the later development of Geometry.

The triangle is introduced not only as a figure but also as a structure, which bears a number of connotations and leads to the conditions of identity and difference that will determine it. Subsequently, a triangle – and as a consequence any complicated geometrical object – would be acceptable if and only if we had rigidity criteria that would make it a structure recognizable in all its potential appearances and permissible transformations.

The logical determination of a geometric figure was set up after the establishment of the "congruence" and "similarity" criteria. We recognize an embodied character in these criteria and apprehend them as potential actions that are advanced in mental acts of comparisons. These universal attributes provide a status for the notion of triangle and would eventually constitute the touchstone ensuring logical comparisons and arguments, (Tall 1995a). Such conditions arise when we introduce new concepts, but also appear in the majority of proofs. Think, for instance, the result concerning the *angles – sum* of a triangle and its variations that determines the intrinsic character of Geometry (Euclidean, Spherical or Hyperbolic).

From the archetypal results to a Logical Theory

The archetypal results constituted a primitive objective knowledge, in a "pre-historical" period of Geometry, before the emergence of any coherent logical[7] deductive theory (this is also indicated by Lakatos, 1997). However in Greek mathematics, a key requirement for geometric investigations was the fundamental (essentially platonic) demand of the anti–visual, as Szabo has noticed:

> *It seems that new kinds of proof appeared at the same time as Greek mathematics was becoming anti-empirical and anti-visual.*
>
> *Arprad Szabo (1978, p. 197)*

The constitution of a Theory for Geometry and the particular need to incorporate the already existing "archetypal" results inside it, consists of a conceptual shift. Apart from certain methodological problems posed in this context, the arising cognitive factors are:

- a demand for formal descriptions inside the Theory and deductive proofs as well;
- the additional derivation of abstract tools (such as the plane, the angle, the area and the construction), since the accumulated empirical knowledge does not suffice;
- Some techniques for constructions and proofs.

[7] *Kneale W. & Kneale M. (1962, p. 2) we have the confirmation "that the notion of demonstration attracted attention first in connection with geometry". A. Szabo has the opinion that the first deductive proof started with Zeno and later went to geometry.*

The plane is intermediate between the sensory intuition and the systematic display of the theory. The plane figures, even in the case they are not congruent, can be compared, as far as the enclosed area is concerned. The constructions are taking place on the plane, a non-trivial mental entity. On the first stage, the involved constructions are actions. Theory demands an overcoming of the particular actions, through verbal descriptions and replacement of them, by mental manipulations. The partial prototype action was transformed into postulates or technical processes where a construction of an object is needed. For example, the embodied action of line segment construction made using pencil and ruler, suggests the first postulate. The construction of such a mental object as the rectangle or the square is determined by recall of partial actions and reasoning as well. On the other hand the proof by superposition is displayed as prototypical one (Tall 1995b).

Furthermore, it is significant to reread a well-known Heath's remark concerning the characterization of a square as a mental instrument in Euclid's Elements and compare it to the production of the figure of an equilateral triangle. Euclid presents the equilateral triangle using "ruler and compass" and he suggests the term Συστήσασθαι (to constitute). We think of this as a condition of constitution, as a metaphor of the constructive action, a fact underlying its immediately enactive (embodied) character. (The same can be said for almost all propositions of the Book I, where the method of congruence is used).

While Euclid demonstrates the existence of a square over a line segment, the term Αναγράψας (which means something described by tracing) is used. We regard this as a reference to logical and symbolic context that came out of the theory and its tools (as axioms, theorems, area etc) and constitutes a proof. Finally, the mental object of a square is assumed in the presentation and the classical proof of the Pythagorean Theorem (Elements I, 46-48).

Pythagorean Theorem

In the case of Pythagorean Theorem, the primitive method of its presentation and explanation via properties of the triads (which may concern either numbers or sides of triangles and arrangements of them) was suitable only in applications of practical problems and in pre-scientific considerations. This was not enough for logical argumentation, which emerged in the context of Geometry. In Proclus' commentary about this result, we trace arguments giving evidence that the general proof of Pythagorean Theorem came out after the shift from figured numbers and dot-squares to *square,* an invented mental instrument. This new perception for square suggested implicitly: an one-to-one correspondence between line segments and squares that would serve as a measure for the enclosed area and permits a multiplication of line segments "regarded as magnitudes" in order to achieve this area.

This situation might be considered as an early twofold representative expression, both of arithmetic and geometric nature, concerning two-dimensional objects. These aspects become dominant in modernity and virtually lead to the algebraic representation of the Theorem via the formula $a^2 + b^2 = c^2$.

The above considerations suggest that the transition from the archetypal formulation to a rigorous geometric presentation demanded not only a high degree of abstraction, intuition and invention, but also a successful work on foundations. Considering the epistemological character of this particular result, we notice that a purely perceptual category such as verticality:

(i) Has been objectified at an early stage, through arithmetical relations. In the orthogonal triangle the discovery of the triads could be raised observing how gravity pulls a weight tied on a thread. The inverse implication (i.e. having a triad we can construct a right angle on a plane) gives the objectification of the idea of right angle independent of the sense of the gravity. We have here a necessary and sufficient condition for the determination of an object.

(ii) Was described completely and precisely by a general mathematical formulation (Pythagorean Theorem), obtaining universal theoretical validity.

Similarity

In ancient Egyptian texts it is cited, under the name of "se-qet" (Heath, 1975, pp. 126 - 128) that the determination of shape is given by means of numerical representations. The geometric significance of these aspects rested upon the fact that angle is the main feature of shape. Furthermore, the plotting of a figure under scale usually preserves angles.

Primal mathematical activities, which traditionally are attributed to Thales, were realized on the ground of angle invariance and obviously they were related to the *similarity of figures*, as we mean it today. This relationship classifies the figures and distinguishes shape, from magnitude. This leads to wider and directly "readable" classes of objects. The theory of similarity is constituted of criteria that explained and established the invariance of the visual shape. We should stress that the objectification of similarity, using the corresponding mathematical notion, would not have been achieved without the development of the Eudoxian theory of proportions. A necessary premise for this process is the shift from relations concerning two figures to internal relations that refer to one and the same figure. The difficulties of early mathematicians in order to obtain the former objectification are justified in the psychological context as well:

> … the origin of the idea of proportions must be sought for in the actual perception of figures… Transposition of the shape entails transposition of the angles but not the relative lengths of sides… One must therefore be careful to avoid thinking that perceptual transposition leads automatically to perception of "similarity" in the geometrical sense.
>
> Piaget, (1956, p. 321)

There is evidence that the ability of recognition of similar shapes among figures in general is an issue that has to do with assimilation of the formal geometrical instruction, Vollrath (1977), Kospentaris & Spyrou (2005), Mattheou & Spyrou (2005).

In the exceptional case of regular polygons the similarity is self-evident: any two regular n-polygons are similar. Of course, such a powerful case is due to structural invariance resulting from symmetry requirements for plane figures.

Another remarkable case of salient similarity is traced back to the writings of Plato, where he intended to describe the form of rigidity:

One of them has at either end of the base the half of the divided right angle, having equal sides, while in the other the right angle is divided into unequal parts, having unequal sides...

Now of the two triangles, the isosceles has one form only; the scalene or unequal – sided has an infinite number…

Of the infinite forms we must again select the most beautiful…

Then let us choose two triangles… one isosceles, the other having the square of the longer side equal three times the square of the lesser side.

(Plato, Timaeus, 53d-54b)

In both of the above cases, Plato achieves a description through the internal ratio of two sides. In the first case he deals with isosceles orthogonal triangles. In the second, he refers to figures derived from the division of an equilateral triangle by its height i.e., orthogonal triangles with angles 30 and 60 degrees[8]. In both cases this self-evident similarity determines the forms.

Considering two triangles, their angles' equality is a consequence of their corresponding side-proportion and vice-versa something which definitely fails for other polygons. As a matter of fact, the apprehension of the general definition of similarity, as it appears (in an admirably complete form) in Euclid's Elements [VI, Definitions 1 and 2], deserves a careful historical investigation. We know that in evaluating such an attempt, inherent epistemological problems are raised, which in Euclid's foundation are hidden behind the formulation of his 5th Postulate.

Discussion and Educational Implications

The main goal of the paper is to understand the theory of embodied mathematics in a genetic way connecting them with the genesis of Geometry. This genetic view is that man builds his mental representation of the world, through a progressive reorganization

[8] A detailed analysis of Plato's geometric ideas, in connection with the above situations, can be found in Popper's writings, in particular see his essay "Plato and Geometry", (Popper, pp. 251-270).

of his prior active manipulation of the environment. Therefore we can determine what embodiment is and which basic functions constitute it. These primitive functions with respect to some specific perceptions mediate to our interaction with reality. With these tools of perception man constitutes the fundamental concepts, analyses and understands the world in an objective way. In particular, we study how the primitive functions of verticality and the recognition of shape are transformed into geometrical notions and mathematical propositions. The guided principle is that, although the very mechanism is based on experience, it could not be reduced solely to bodily basis. The biological subjects participate in historical events and interweave into the origins of Geometry (Husserl, 1998). Therefore, we provide a view in bringing historical contexts in our analysis, focusing on the Greek period of Mathematics. In that period some intellectual demands, such as the anti visual and the dialectic lead to an abstraction which encapsulates the empirical knowledge in a theory.

We have done some subtle distinctions to the current cognitive tools, especially that of prototypes. The proposal consists of consolidation and conscious manipulation of the distinctions of elementary prototypes in the case of points, lines and planes, compound prototypes (the case of the angles), patterns (such as triangles) and enactive prototypes for the constructions and proofs.

A successful presentation of the above interconnection can be considered useful for the initial teaching of geometry. Some specific aspects were pointed out, the embodied feature of which can shed light to their place in Euclidean Geometry. This can be recognised in definitions, axioms, constructions and finally in archetypal results.

A main question for mathematics education is in which way a teaching model that takes care of current psychological and educational views can be formed. The connection of the concepts of gravity with the notion of verticality is necessary. Next, we will discuss the representation of Pythagorean triads as a tool of characterisation of verticality. In order to teach Pythagorean Theorem a passing through triads, as figured dotted squares is helpful, before we should present the proof of it, using the notion of the area. The classic formulation of Pythagorean Theorem combines integrally angles, constructions of objects, areas and implicitly constitutional principles as the theory of parallel lines. All of them consist of a reasoning approach to proofs and reveal not only the archetypal character of the result but also a meaning of what is objective in two different contexts that of pre-geometric and that of geometric.

The presentation of similarity uses the form of angles in the case of triangles. More subtle remarks are included in the distinction of equality and an accepted notion of similarity that overcome the case of automatic similarity. It is noteworthy that in order to consider similarity, the reduction to prototypes does not suffice, since in the full study of the concept the proportion enters.

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