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SEMANTIC AND SYNTACTIC PROOF PRODUCTIONS

ABSTRACT. In this paper, we distinguish between two ways that an individual can construct a formal proof. We define a syntactic proof production to occur when the prover draws inferences by manipulating symbolic formulae in a logically permissible way. We define a semantic proof production to occur when the prover uses instantiations of mathematical concepts to guide the formal inferences that he or she draws. We present two independent exploratory case studies from group theory and real analysis that illustrate both types of proofs. We conclude by discussing what types of concept understanding are required for each type of proof production and by illustrating the weaknesses of syntactic proof productions.

KEY WORDS: abstract algebra, advanced mathematical thinking, advanced mathematical concepts, convergent sequence, formal reasoning, isomorphism, limits, proofs, real analysis

1. INTRODUCTION

Reasoning about advanced mathematical concepts involves a complex interaction between rigorous and intuitive thought. Advanced mathematical concepts differ from those encountered earlier in that they are determined by a precise, unambiguous definition, usually expressed using logical symbols. Formal reasoning about such concepts requires the use of this definition, and the inferences that one can draw are limited to a set of well-defined, agreed upon procedures (Tall, 1989). While a high level of rigour is required when reasoning about mathematical concepts, it has been argued that one also needs intuitive, non-formal representations of these concepts to reason about them effectively. For instance, Fields Medalist William Thurston declares that learning about a mathematical topic consists of building useful non-formal mental models, and that this cannot be accomplished by studying definitions and rigorous proofs alone (Thurston, 1994). Fischbein (1982) argues that removing intuitive representations from formal mathematical thought is neither possible nor desirable, as these are necessary components of productive mathematical thinking. Eminent mathematicians have testified to the importance of intuitive thought in their own mathematical work (e.g. Hadamard, 1945; Poincaré, 1913). On a less positive note, students' intuitive understandings of advanced mathematical concepts have been implicated in the errors they make (relative to the formal theory) in reasoning about these concepts (e.g. Tall and Vinner, 1981; Vinner,

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1991), indicating that this intuition has a significant role in their reasoning too.

Our purpose in this paper is to further clarify the meanings of 'formal' and 'intuitive' reasoning, the role each can play in producing proofs within a given area of mathematics, and the ways in which they are related. We accomplish this by discussing two qualitatively different ways in which an individual might produce a correct proof, and illustrating these by using data from independent studies in group theory and real analysis.

2. Syntactic and Semantic Proof Productions

When asked to prove a statement, professional mathematicians and logically capable mathematics students all share the same goal – to produce a logically valid argument that concludes with the statement to be proven. However, the processes used to complete this goal may vary widely. On one extreme, students might construct a proof by reproducing an argument that has been memorised by rote, or applying an algorithm that they have been told will produce a valid proof. Proofs of this type require minimal engagement on the part of the prover and the prover may be unaware of how his or her proof establishes the veracity of a mathematical statement. At the other extreme, some proofs may involve the prover constructing new mathematical concepts, or re-conceptualising old concepts in radically different ways. In this paper, we wish to describe two types of proof productions that lie between the two extremes described above. More specifically, we distinguish between proof productions in which the prover works from a literal reading of the involved definitions and proof productions in which the prover also makes use of his or her intuitive understanding of the concepts.

We define a *syntactic proof production* as one which is written solely by manipulating correctly stated definitions and other relevant facts in a logically permissible way. In a syntactic proof production, the prover does not make use of diagrams or other intuitive and non-formal representations of mathematical concepts. In the mathematics community, a syntactic proof production can be colloquially defined as a proof in which all one does is 'unwrap the definitions' and 'push symbols'.

We define a *semantic proof production* to be a proof of a statement in which the prover uses instantiation(s) of the mathematical object(s) to which the statement applies to suggest and guide the formal inferences that he or she draws. By an instantiation, we refer to a systematically repeatable way that an individual thinks about a mathematical object, which is internally meaningful to that individual. (We prefer instantiation to the more general term representation since we wish to exclude cases in which an individual represents a mathematical concept by rephrasing its definition or with an inscription to which they cannot attach meaning). For example, an instantiation of a particular sequence might consist of the sequence plotted on a Cartesian graph or of the terms of a sequence listed as an infinite string of numbers. An instantiation of an arbitrary convergent sequence might consist of a graph of a 'prototypical' convergent sequence. During a semantic proof production, the prover may physically draw or write these instantiations or he or she may work with them mentally. What is crucial is that the prover use these instantiations in a meaningful way to make sense of the statement to be proven and to suggest formal inferences that could be drawn.

In the two sections that follow, we present independent exploratory studies from group theory and analysis that serve both to illustrate and further refine these theoretical constructs. Although we believe that while syntactic proof productions can be considerably valuable in some circumstances, when presenting our studies, we also hope to highlight some important limitations of these proof productions. In particular, we hope to show that the number of proofs that can be produced in this way may be relatively limited and that after a syntactic proof production, the prover may feel somewhat unconvinced that the proven claim has been established.

3. Exploratory studies about reasoning with group isomorphisms

In the previously published research, Weber (2001) found that undergraduates were often unable to prove or disprove that two groups were isomorphic, even in cases when they possessed the factual knowledge and procedural proficiency to do so. In a separate line of research, Leron et al. (1995) demonstrate that even if an undergraduate could give an accurate formal description of isomorphic groups, his or her intuitive understanding of this concept could still be weak, and he or she might hold inaccurate beliefs about this concept.

The exploratory studies presented below extend this previous research, and are presented here to highlight the contrast between syntactic and semantic proof production in the work of undergraduates, doctoral students and professional algebraists. In the first study, we take an in-depth look at undergraduates' and advanced doctoral students' behaviours while attempting to prove or disprove that pairs of groups were isomorphic. The results of this study indicate that the undergraduates attempted to use restricted syntactic reasoning only, while the doctoral students appeared to use a more global view of certain groups to guide their production of syntactically correct arguments. In the second study, we investigate undergraduates' and algebraists' ability and inclination to instantiate groups. This allows us to be more precise about how each population used such instantiations when producing correct formal arguments.

3.1. *A study of students' attempts to prove statements about group isomorphisms*

3.1.1. *Methods*

Participants. Two groups of students participated in this study. The first consisted of four undergraduates at a university in the northeastern United States. These students had recently completed their first abstract algebra course. Each student had also completed two linear algebra courses, the second of which stressed abstract vector spaces and rigorous proofs. The second group of participants consisted of four doctoral students completing dissertations in an algebraic topic at a university in the midwestern United States. These students had approximately four more years of mathematics than the undergraduate students.

Materials. Participants were first asked to prove the propositions presented in Table I.

Table I
Propositions used in isomorphism study

B1. Let G and H be groups and f be a homomorphism from G to H. Prove that for all x and y in G, $[f(xy)]^{-1} = f(y^{-1})f(x^{-1})$.

B2. G is a group and f

The first two propositions, B1 and B2, were basic propositions and were included to determine whether the participants possessed an ability to construct rudimentary proofs. The last five propositions ask the students to prove or disprove that a pair of groups is isomorphic.

Procedure. Participants were asked to 'think aloud' as they attempted to prove the propositions listed above. At any point, the participants were allowed to refer to the textbook used in the undergraduate abstract algebra course.

After attempting to prove the propositions, the participants completed a paper-and-pencil test about the facts needed to prove the propositions in this study. This test contained free response questions (e.g., 'State the definition of isomorphic groups') as well as yes-or-no questions (e.g., 'Can an abelian group be isomorphic to a non-abelian group?'). After each question, the participants were asked to indicate the degree of confidence they had in their answer with an integer between 0 and 2, where 0 represented 'just guessing', 1 represented 'fairly certain', and 2 represented 'absolutely certain'.

If participants had been previously unable to prove a proposition, they were invited to try again by making use of their work on the paper-and-pencil test.

3.1.2. *Results*

All participants in this study could prove the two basic propositions (B1 and B2) indicating that they had a basic notion of proof, familiarity with group-theoretic concepts, and some ability to manipulate logical symbols. In most cases, the undergraduate students could also identify properties satisfied by the groups under consideration. For instance, all undergraduates were aware that $\mathbf{Z}$ and $\mathbf{Z}_n$ were cyclic and abelian groups.

Each doctoral student was able to prove all five Isomorphism Propositions (I1 through I5 in Table I). In contrast, the undergraduates collectively proved only two of the 20 Isomorphism Propositions. Our analysis indicates that there were nine instances in which an undergraduate failed to construct a proof despite appearing to possess the factual knowledge required to do so. To be specific, in these cases, the undergraduate was not at first able to construct a proof. However, on the paper-and-pencil test, the undergraduate correctly answered each question about the facts needed to prove the proposition in question with a degree of confidence of 1 or 2. After completing this test, the undergraduate was able to prove the proposition using these facts. Hence, the data replicate the results of a similarly designed study on group homomorphisms (Weber, 2001) indicating that even if one has an accurate conception of proof, possessing a factual knowledge of a mathematical concept does not imply that one can effectively prove statements about that concept.

To investigate the reasons for the discrepancies between the performance of doctoral students and undergraduates, we analyse the behaviour of the participants while they are attempting to prove three of the propositions. An analysis of the other two propositions is given in Weber (2002b).

*Prove or disprove **Q** is isomorphic to **Z**.* The protocol of one doctoral student's proof is given below:

Z is isomorphic to **Q**? That's false. Let's see ... why? Well **Q** is dense and **Z** is not. No wait, denseness isn't a group property. Well then **Z** is cyclic and **Q** is not. So they can't be isomorphic.

Two other doctoral students proved that the groups could not be isomorphic because **Z** was cyclic and **Q** was not, with one adding, 'I was tempted to add something about **Q** having a field structure, but that's not really the point'.

The following excerpt of one undergraduate's protocol is given below:

Um I think that **Q** and **Z** have different cardinalities so ... no wait, **R** has a different cardinality, **Q** doesn't. Well, I guess we'll just use that as a proof. Yeah so I remember like seeing this proof on the board. I just don't remember what it is. There's something about being able to form a uh homomorphism by just counting diagonally [the student proceeds to create a complicated bijection between **Z** and **Q** by using a Cantorian diagonalization argument] Yeah I don't think we're on the right track here. Um ... what you are describing is ... it's um a bijection, but not a homomorphism.

This excerpt was representative of all four undergraduates' proof attempts. Upon realising that the integers and the rational numbers were equinumerous, all undergraduates constructed or attempted to construct a bijection between the groups using a complicated Cantorian diagonalisation argument. It appears that they were sidetracked by a particular bijection that they encountered previously, and they seemingly showed little regard as to whether their bijections would respect the groups' operations. None successfully proved that the groups were not isomorphic.

Prove or disprove $\mathbf{Z}_p \times \mathbf{Z}_q$ is isomorphic to $\mathbf{Z}_{pq}$ (assuming p and q are coprime). An excerpt from one doctoral student's proof attempt is given below:

OK, sufficient to find an element (g, h) in $\mathbf{Z}_\mathrm{p}$ times $\mathbf{Z}_\mathrm{q}$ that has order pq, because $\mathbf{Z}_\mathrm{p}$ times $\mathbf{Z}_\mathrm{q}$ has order pq and so if there's an element with the same order as the group, the group is cyclic and *must be the same group* as $\mathbf{Z}_\mathrm{pq}$. OK um the element we're looking for is going to be (1, 1). [italics are our emphasis]

The student then proceeded to show (1, 1) had order pq. The other doctoral students all proved that these groups were isomorphic by first observing that equinumerous cyclic groups were isomorphic and then showing that $\mathbf{Z}_p \times \mathbf{Z}_q$ was cyclic. No doctoral student constructed an explicit isomorphism between the two groups.

The two undergraduates who made progress on this problem attempted to construct a bijection between the two groups, one of which was the idiosyncratic mapping that mapped (a, b) in $\mathbf{Z}_p \times \mathbf{Z}_q$ to ab (mod pq) in $\mathbf{Z}_{pq}$. This mapping was neither bijective nor a homomorphism. Neither of these undergraduates used the fact that p and q were coprime. The other two undergraduates did not know how to begin their proof attempts.

Prove or disprove S_4 is isomorphic to D_{12}. All of the undergraduates in this study had a very limited knowledge of the dihedral groups, and none was able to make substantial progress on this problem. Upon noting that S_4 and D_{12} were both equinumerous, non-abelian groups, the doctoral students attempted to locate a distinct property of one group and demonstrate that the other group did not share this property. In two cases, after locating such a property the doctoral student proved that two groups could not be isomorphic if one had this property and the other did not. One of these two doctoral students used diagrams to aid his reasoning. He recognised that D_{12} represented 'symmetries of a 12-gon' and drew a regular 12-sided polygon. He then used this diagram to recognise that D_{12} would have elements of order 12 and then showed that S_4 did not.

3.1.3. *Discussion and analysis of data*

We analysed and categorised each of the undergraduates' and doctoral students' proof attempts. We present the results of our analysis in Table II.

In most of the cases where the undergraduates seriously attempted to prove an Isomorphism Proposition, their proof attempt was of the following form: Upon realising that the groups in question were equinumerous, they

Table II
Classification of undergraduates' and doctoral students' proofs

Type of proof attempt	Undergraduates' proof attempts	Doctoral students' proof attempts
Compared groups' properties	1 (1)	18 (18)
Examined mappings between the groups	9 (0)	1 (1)
Looked for counter-example of general statement	1 (1)	1 (1)
Unable to make meaningful progress	9 (0)	0 (0)

Successful proof attempts of each type are given in parentheses.

attempted to construct a (seemingly arbitrary) bijection between the groups. If this construction was successful, they were dismayed to find that the bijection did not respect the groups' operations and abandoned their proof attempts. If the construction was unsuccessful, they also gave up as they did not know how to proceed. There were nine proof attempts in which undergraduates examined mappings between the groups; in each case, the undergraduate failed to produce a proof. It is notable that undergraduates rarely employed properties (other than the order of the groups) that they knew to hold for the groups in question. For instance, on the paper-and-pencil test administered after the undergraduates attempted the proofs, all of the undergraduates indicated that they knew **Z** was cyclic, **Q** was not, and a cyclic group could not be isomorphic to a non-cyclic group. Yet no one considered these facts in their initial proof attempts and none successfully proved that **Z** and **Q** were not isomorphic at this stage. What they seem to lack is an inclination to introduce these properties spontaneously, or alternatively, perhaps, a mechanism for deciding which could be productively introduced.

The behaviour of the doctoral students' was considerably different. The doctoral students seldom employed the definition of isomorphic groups; in fact there was only one instance in this study where a doctoral student made any mention of an explicit mapping between the groups with which he was working. The doctoral students seemed quite consistent with their proof attempts: they either attempted to show that the two groups shared properties that ensured the two groups were necessarily isomorphic or they tried to show one group held a property that the other group did not share. To prove $\mathbf{Z}_p \times \mathbf{Z}_q$ was isomorphic to $\mathbf{Z}_{pq}$ when p and q were coprime, the doctoral students did not attempt to construct an isomorphism between the groups; rather they tried to show the groups shared crucial properties – that they were equinumerous cyclic groups. When the groups were not isomorphic, the doctoral students almost always attempted to find a property that one group possessed and the other did not.

3.2. *An investigation of undergraduates' and algebraists' intuitive understanding of group isomorphism*

3.2.1. *Motivation for this study*

After analysing the data reported in the last sub-section, we conjectured that the doctoral students understood isomorphic groups as being 'essentially the same' and were using this understanding, together with knowledge of the groups in question, to guide their formal work. In contrast, the undergraduates seemed unable to use their knowledge about particular groups in such an effective way. We further conjectured that the undergraduates were not using, and quite likely did not possess, a comparable intuitive understanding of isomorphism and a mechanism for deciding which properties would be relevant. However, while our data do support this conclusion, they are also consistent with other alternative hypotheses. For instance, it is also possible that both groups understood group isomorphisms at a purely formal level and the doctoral students simply chose to compare the groups' structural properties because experience had shown this to be a more efficient problem-solving approach. (Indeed, Weber (2001) drew exactly this conclusion.) The study reported in this sub-section was designed to consider such rival hypotheses by shedding light on the following questions: What are undergraduates and algebraists' formal and intuitive understandings of isomorphic groups? How are these formal and intuitive understandings related? How are they each used when proving or disproving that two groups are isomorphic?

3.2.2. *Methods*

Participants. Two groups participated in this study. The first consisted of four undergraduate students at a university in the southern United States. These students had recently completed their second course in abstract algebra. Topics in this course included the first isomorphism theorem in group theory, as well as ring and field isomorphisms. The second group consisted of four mathematics professors working at the same university. All held Ph.D.'s and all regularly used group-theoretic concepts in their research.

Procedure. Each participant met the first author individually for a semi-structured interview. The participants were asked each of the questions presented in Table III. If participants made a response that the interviewer

Table III
Questions asked in isomorphism interview

- Describe for me all the ways that you represent or think about algebraic groups.
- What does it formally mean for two groups to be isomorphic?
- Describe for me in your own words what it means intuitively for two groups to be isomorphic?
- How is the intuitive description that you just gave related to the formal definition of isomorphic groups?
- Suppose that I asked you to prove or disprove that two groups were isomorphic. Describe for me some of the ways that you would approach this problem.
- Would you ever use your intuitive understanding of isomorphic groups to try to construct this type of proof? If yes, how so?
- (If the answer to the previous question was yes) Proofs are supposed to be formal, yet you gave an intuitive approach for writing these types of proofs. How can your intuition be used to create a formal argument?

found interesting, he would ask questions to further probe the participant's understanding. When participants used a vague or ambiguous phrase in their answer, they were asked to clarify.

3.2.3. *Algebraists' responses*

Algebraists' representations of groups. The four algebraists all provided multiple ways in which they could represent groups. Three indicated that they thought of groups in terms of their group multiplication tables and three stated that they thought of groups abstractly in terms of generators and relations. There were also many other representations of a group offered only by a single algebraist. Such representations included a directed graph, symmetries of a geometrical object, and a Venn-like diagram of a set partitioned by a sub-group and its cosets. All algebraists mentioned that they had representations that only applied to specific groups, such as matrix groups. In the terms used in this paper, all the algebraists were therefore able to instantiate groups in various ways.

Algebraists' understanding of isomorphic groups. Each algebraist provided two intuitive descriptions of what it meant for two groups to be isomorphic. First, each indicated that he thought of isomorphic groups as being algebraically the same. The specific phrases the algebraists used were 'essentially the same', 'exactly the same structure and exactly the same form', 'algebraically the same', and 'the same group except the elements and operation may have different names'. The other response was that two groups were isomorphic if one group was simply a re-labelling of the other group. In the words of one algebraist,

> In some sense, I should be able to line up all the elements of one group and ... index rows and columns in some multiplication table and for one group and line the elements for the other group up in some type of table so that when I look at them, the only thing that is different is the particular symbol that is representing the element, but otherwise I'm looking at the same table. That to me would be another picture of isomorphic groups.

We note that this global notion of 'essential sameness' appears to be described in terms of at least one imagined instantiation (i.e. the groups' multiplication tables) of two isomorphic groups.

The algebraists were also able to describe the way in which these 'global' notions based on comparing instantiations translated into 'formal' expressions based on use of definitions. When asked how their intuitive descriptions of isomorphic groups related to the formal notions, all algebraists noted that isomorphic groups are re-labellings of each other because the isomorphism serves as the re-labelling. As one algebraist describes, In the formal definition, this bijective mapping is what's giving you this correspondence between elements in the two tables. That's what the bijection's doing. This mapping is really locating corresponding elements in the table and making sure products of elements go the appropriate place. [As he is saying this, the mathematician is making hand motions in the air relating elements from one imaginary group multiplication table to another]. That's sort of hard to visualize – I'm doing hand motions in the air.

Three algebraists noted that isomorphic groups are algebraically the same because an isomorphism will preserve all algebraic properties. In the words of one algebraist,

An algebraic property can usually be written as a logical sentence involving arbitrary group elements and the group operation. It's not hard to show that a bijective homomorphism will always preserve this type of property.

In sum, the algebraists were able to give multiple ways of instantiating groups and descriptions of how the notion of these having 'the same structure', as embodied in the instantiation, could be translated into formal terms using the definition of isomorphism.

How algebraists would prove or disprove whether two groups were isomorphic. Each of the algebraists indicated that they would begin their explorations by obtaining a qualitative or intuitive feel for each of the groups. The algebraists' responses were vague; in their words, they said they would 'size up the groups', 'get a feel for the groups', 'look at the groups', or 'see what they [the groups] looked like'. When the interviewer asked them to clarify what they meant, the algebraists were generally unable to do so.

Two of the algebraists also explicitly mentioned that they would *not* examine mappings between the groups, due to the fact that such an approach was impractical if the groups under consideration had even a moderately large order. All algebraists also added that to show that the two groups were isomorphic, they would generally attempt to find properties shared by each group that ensured such an isomorphism would exist. It appeared from their comments that this was achieved by using a combination of instantiation of groups as group multiplication tables, generators and relations, and so on, and factual knowledge of the groups' distinguishing properties. (e.g., $\mathbf{Z}_n$ is *the* cyclic group of order n).

Two algebraists again offered clear frameworks for how their intuitions could be translated into formal mathematical language. In one algebraist's words:

If I can find a way in which the groups are different, I can always . . . um I should say I can usually write this as statements about operations applied to group elements. You know, a group is commutative if $xy = yx$ for all elements in the group, it's very easy to show using the definition of isomorphism, that if one group has that property, the other does too. It's just a matter of expressing the property that just one group has–be it cyclic, commutative, simple, whatever–as a formal statement about elements in the group.

The two other algebraists did not offer specific methods for how their intuition directly related to their formal arguments, but claimed (in one algebraist's words) that 'any intuitive insight would fairly quickly lead to a formal demonstration'. Their comments were thus similar to the comments of a professor in Raman's (2003) study when she asked a mathematician to prove that the derivative of an even function was odd.

Overall, the algebraists had multiple ways of instantiating groups, an understanding of isomorphism as 'essential sameness' that was tied to these instantiations, knowledge of the properties of certain groups, and an ability to describe the way in which the formal definition of isomorphism could be used to show that an isomorphism would preserve such properties.

3.2.4. *Undergraduates' responses*

Undergraduates' representations of groups. Perhaps surprisingly, none of the undergraduates was able to provide a single representation of a group other than a structure that satisfies a list of formal axioms. When asked how they represented a group, all undergraduates stated a group's formal definition. Further questions by the interviewer did not lead any of the undergraduates to elaborate on their responses.

In this respect, the undergraduates' responses are consistent with Hazzan's (1999) observation that undergraduates often have difficulty building concrete representations of abstract concepts from group theory.

Undergraduates' understanding of isomorphic groups. All four undergraduates gave the formal definition of isomorphic groups, but no undergraduate could provide an intuitive description of what it meant for two groups to be isomorphic. One undergraduate did make a revealing response when she claimed, 'my intuition and formal understanding of isomorphic groups are the same'.

How undergraduates would prove or disprove whether two groups were isomorphic. The undergraduates were uniform in their responses. Each said that they would first compare the order of the two groups. If the groups had the same order, they would look at bijective mappings between the two groups and see if they were isomorphisms. The undergraduates claimed that they would not use their intuition to determine whether the two groups were isomorphic.

3.3. *Relationship between the two studies*

The undergraduates in the second study indicated that they would attempt to syntactically prove or disprove that the two groups were isomorphic. As the undergraduates' behaviour in the first study was quite consistent with the undergraduates' responses from the second study, this suggests that the first participants also were attempting syntactic proof productions. There were (at least) two factors that may have prevented the undergraduates in the second study (and possibly the first) from producing proofs semantically. First, the undergraduates viewed the concept of isomorphic groups as a purely formal construct, and appeared to lack any notion of thinking of the two groups as being isomorphic if they were essentially the same. Second, they also appeared to lack the ability to instantiate the groups in a meaningful way, making it difficult for them to make global qualitative comparisons between groups.

The algebraists in the second study indicated that they would try to semantically prove or disprove that the two groups were isomorphic. When asked how they would approach such a task, they all indicated that they would first try to get a qualitative feel for the two groups in question and then would attempt to determine whether the two groups were essentially the same. From the data presented in the first study, we cannot conclusively determine whether the doctoral students were attempting syntactic or semantic proof productions. However, there are three factors that lead us to conjecture that at least some of their proof productions were semantic. First, the doctoral students' behaviour was consistent with the algebraists' responses. As the latter group indicated that they would attempt to produce semantic proofs, this suggests to us that the doctoral students may also have been producing their proofs semantically. Second, there were several instances in which the doctoral students spoke of isomorphic groups as 'being the same group', suggesting that this intuitive notion of isomorphic groups was influencing their reasoning. Third, there was one instance where a doctoral student drew a 12-gon as an instantiation of the group D_{12} and used this diagram to identify a crucial property of this group. This, to us, was an unambiguous instance of a semantic proof production.

4. A CASE STUDY OF TWO STUDENTS' PRODUCTION OF A PROOF ABOUT CONVERGENT SEQUENCES

The case study reported in this section was taken from a larger qualitative study of the way that different learning environments influence students' developing an understanding of real analysis (Alcock and Simpson, 2002). In this larger study, students were interviewed fortnightly in pairs for the duration of their first 10-week course on this subject. During these interviews, students answered questions designed to probe their understanding of the concepts involved, both directly and through asking them to evaluate and prove results about these. From these interviews, an unanticipated finding emerged. It became clear to us that students were reasoning about the concepts in real analysis in qualitatively different ways. Some relied on rich visual images to aid in their formal mathematical reasoning. Others seemed to work with concepts as purely formal constructs, the properties associated with which could be manipulated according to the rules of first-order logic. In this section, we will examine one pair of students, each of whom reasoned about the concept of convergent sequences in one of these ways. In particular, we will focus on these students' reflections on their understanding and reasoning.

4.1. *Methods*

Participants. The students introduced in this section are Adam and Ben, who attended the fortnightly interviews together. Both were attending a classroom-based course in real analysis based on R.P. Burn's book 'Numbers and Functions: Steps into Analysis' (Burn, 1992), which was taught at a prestigious university in the UK. In this course, students worked together to answer a structured sequence of questions and thereby proved most of the results of the course themselves (more detail can be found in Alcock and Simpson, 2001). The course covered work on sequences, completeness of the real numbers, and series, all on a formal basis using standard $\varepsilon - N$ definitions for convergence. This was the first time the students had encountered mathematics based on formal definitions, although all of those attending had very high grades in their previous mathematical work.

Both Adam and Ben were among the most successful of the 18 participants in the study. Adam, however, regularly used visual images in explaining his reasoning, whereas Ben worked solely with the formal constructs.

Materials. Our analysis in this section is based upon the task presented in Table IV, which was assigned to the pairs of students during the fourth interview in the seventh week of the course.

Procedure. In all task-based sections, the students were encouraged to confer with each other and to use the paper on which the question appeared for writing anything that might be useful. During this time the interviewer's interventions were minimal, and questions directed to the interviewer were generally deflected back to the students. Once the students were satisfied

Table IV
Task set for students in week 7 of analysis study

Consider a sequence (a_n). Which of the following is true?
(a) (a_n) is bounded $\Rightarrow$ (a_n) is convergent,
(b) (a_n) is convergent $\Rightarrow$ (a_n) is bounded,
(c) (a_n) is convergent $\Leftrightarrow$ (a_n) is bounded,
(d) none of the above.
Justify your answer.

with their answer, the interviewer instigated a conversation about the strategies the students had used and about definitions and proof in general.

4.2. *Adam and Ben's proof that convergent sequences are bounded*

After reading the question, Adam and Ben quickly chose (b) as the correct statement, with Adam leading in instantiating counter-examples to the others. Following this, they promptly outlined an argument, largely due to Ben's immediate introduction of the appropriate definition:

B: Right, (b) . . . is convergent. Right, our definition of convergence is that . . . well, there exists an N such that when n is greater than big N, there . . . modulus of a is less than epsilon. So that leads to epsilon being a bound . . . plus or minus epsilon being a bound, about a. a_{n}. Is it?

A: Yes go on do you want to write that down? Right the . . . you have it bounded by, the plus or minus epsilon thing, and that shows it's eventually bounded and then it is . . . bounded. By sort of, an earlier result, sort of thing.

Ben did some writing, and was asked to read out his work. At this point Adam quickly made minor corrections to Ben's work.

B: It's only sort of, vague. But, modulus of a_{n} will be less than epsilon if it's converging, when n is greater than big N. Hence minus epsilon is less than a_{n}, is less than epsilon, therefore epsilon is a bound. Since a_{n} is eventually bounded, therefore a_{n} is bounded by . . . whatever various proofs that we've done in the book!

I: Mm. What were you going to say, Adam?

A: If it's convergent, rather than converging or tending to zero . . . that, the modulus of a_{n} you've written, should be modulus of, a_{n} minus a. And you have to say, "if, the sequence a_{n} tends to a, then" . . . and that needs to be the modulus of a_{n} minus a

B: Yep.

A: Is less than epsilon.

Following this, Adam and Ben outlined an almost complete proof, which Ben explains as follows.

B: Alright, so you start off just to prove that if, a series is–a sequence is eventually bounded then ... the sequence is bounded. Which is ... what basically ... yes, you're assuming that a_n is eventually bounded so it's got some limits, and then there's a fixed number of values, before those limits, so before big N. And then, the upper bound is the maximum value of, a_1 to a-big-N, and the upper bound of the eventually bounded part. Then its er ... lower bound is the minimum of a_1 to a-big-N, and the lower bound of the eventually bounded part. Then that's used for the second part, which is er ....basically er ... when, a_n is ... a_n's convergent a_n is bounded, which, is proved by ... a_n is converging to a, then a_n minus a, no for n bigger than big N, a_n minus a–modulus of a_n minus a is less than epsilon. And er ... so erm ... then you have a minus epsilon, is less than, a_n–no ... yes, a minus epsilon, is less than a_n, is less than a plus epsilon, making the two bounds? And, because, that occurs when n is greater than big N, then that means, a_n is eventually bounded, and so by the previous, sort of proof, we've shown that, if a_n is eventually bounded then it's bounded. So a_n's bounded.

Ben's speech is hesitant, and he misuses the term 'limit' in this technical context. However, the last excerpt illustrates that he understands the logical structure of the argument, and is able to argue with the definition of convergence in this way.

4.3. *The students' reflections on their use of definitions*

Following the above proof production, the interviewer initiated a discussion about the strategies that the students used.

I: So at the beginning you started off, the first thing you wrote was something to do with the definition, yes?

B: Yes.

I: Did ... is that because you remember doing this particular proof, or is that a tactic that you use in general?

B: I ... they always say, start off with the definition if you're stuck.

I: Yes.

B: So, started off and see what happens from there.

I: Yes. Does that work, usually?

B: Erm, well it's worked this time!

From this excerpt, Ben apparently begins some of his proof attempts by introducing relevant definitions and working from these. In his response below, Adam adds that he believes that this in itself is often insufficient.

A: *It's not usually enough to stick the definition down*, you have to stick it down and *then remind yourself of what it means*.

I: Yes?

A: *Because the definition and how you understand it are never like, exactly the same*.

I: Are they not? Can you explain to me what the difference is Adam?

A: I really wish I hadn't said that now! Er, yes it's like . . . you understand that it gets, closer and closer,

I: Yes,

A: But you can't just . . . the definition – you can't put it gets closer and closer, you have to have the "for all epsilon greater than zero, n greater than N implies, modulus of a_n, minus a", and all that sort of stuff. It's got to be put into the mathematical terms. [our italics throughout]

Adam's comments indicate that he sees a difference between the formal definition of convergence and how he understands the concept. At this point, the interviewer asked specifically how Adam understood the definition of convergence.

I: Yes. Do you feel you understand that definition though?

A: Yes.

I: Yes. Can you explain to me, what that means in words, the definition?

A: That it's, getting closer and closer all the time . . . I can't do it without (gestures) . . .

I: Yes, no, draw! Write, draw, please . . .

A: No it's my finger I can do it with my fingers . . . !

I: Oh with your fingers, all right then.

A: Which is really helpful! It's getting closer and closer . . . there's, if you pick a limit, of some distance away from – limit was the wrong word to use. If you pick a value sort of some distance away from . . . the value which it converges to, there's a point after which it's always . . .

I: Mm.

A: Between that.

When asked about his view of what Adam has said, Ben responds that he does not see a similar contrast between the formal definition of convergence and his way of thinking about the concept.

I: Do you find there's a bit of a difference Ben, between how you think about convergence and what the definition says?

B: Not really no.

I: Not really. Did you to begin with?

B: Erm . . . to begin with I didn't understand convergence at all. I thought okay, and just accepted it and then . . . now it's just become a sort of natural way of thinking about it.

Adam concluded this section of the interview by adding a final comment on the importance of considering the meaning as well as the form of the definition when he is writing proofs.

A: . . . I still have to, sort of like . . . because you can just write down the definition, without, kind of remembering what it means, I find. So, you have to like, sort of write it, and think like . . . what does that mean, and what have I got, and how can I put it into that form?

Overall, Ben appears to think about convergence solely in terms of its formal definition. We found it interesting that he claims not to use any pre-conceived notions of limit; this contrasts with much of the literature on limits and convergence, which indicates that it is more common for students to have a strong prior sense of understanding the concept, often incorporating spontaneous conceptions that are based on the everyday use of the terms (Cornu, 1992; Monaghan, 1991). Ben's way of thinking is consistent with his approach to writing proofs about convergence. When asked about his immediate introduction of the definition in the preceding proof, Ben reports that he 'started off with the definition' and 'saw what happened from there'. In particular, Ben gave no indication that he referred to instantiations of convergent sequences or any other intuitive understanding of the concept in his formal reasoning. In our view, Ben's actions are consistent with an individual who is engaged in a syntactic proof production.

Adam, on the other hand, reports that he thinks about convergence both in terms of its definition and using other intuitive representations. Unfortunately, from these excerpts, we cannot conclusively determine exactly how Adam is instantiating this concept, although his gesticulations are consistent with the standard graphical representation of the limit of a sequence. He does stress the need to use intuitive representations while he is constructing proofs, and makes a specific point about translating between the formal definition and his other representations in saying that his proof-writing involves thinking about the meaning of the definition and formulating his thoughts in similar language. As Adam reports that his intuitive representations of convergence play an important part in his proof-writing, we would say that Adam engages in semantic proof productions.

One final issue that we would like to discuss is the degree of conviction that Adam and Ben appear to obtain from producing proofs in their respective ways. Ben's comments transcribed below were taken from earlier in the week 5 interview, and were typical of a number of comments that he made throughout the course.

B: . . . I understand how to sort of . . . use it [a definition] to my advantage, what I'm told.

I: Right.

B: But . . . I don't actually understand the concept fully as such. I . . . I suppose it just takes some time.

I: Can you give me an example?

B: Erm, sort of like . . . when it's converge – null sequences converging, [note: null sequences were defined to be sequences that converged to the number zero].

I: Mm,

B: Erm . . . before I just couldn't have the faintest idea of how when a number is less than epsilon,

I: Okay,

B: That shows that . . .. Now I know, okay, if you want to show a null sequence converges then you have to do that,

I: Yes,

B: But, I'm still not quite on top of that.

Ben expresses concerns about his own learning, in particular about the amount of time it takes to 'really understand' things. It appears that Ben desires an internally meaningful understanding of the concept of convergence, but is failing to reach this, at least in the short term. Ben knows what needs to be done to produce a proof of a statement, but seems unsure of why his arguments establish the proven claim. By contrast, Adam expresses no such reservations; he speaks of his need and ability to attach meaning to his formal work.

4.4. *Conclusions from this study*

Due to the nature of these data, we cannot conclusively determine whether Ben would have been able to produce the proof without Adam's assistance (or, for that matter, whether Adam would have been able to produce the proof without Ben). However, both students achieved a high level of success on subsequent examinations in which they worked independently. Despite this, Adam and Ben's understandings of the concept of limit were qualitatively different. Adam viewed the definition as being intimately linked with instantiations of sequences and he used these in conjunction with his formal reasoning. Ben did not have such a means of thinking about the 'meaning' of terms such as *convergent sequence* and his reasoning consisted of symbolically manipulating the definition. Although they could both produce formal proofs, Adam appeared more comfortable with his understanding of the topic.

5. Integration of the Two Exploratory Studies

In analysing the data presented in the previous two sections, it became clear to us that the participants attempted to produce proofs in each of their respective subject areas (group theory and real analysis) in qualitatively different ways. The undergraduates, in Section 2, and Ben, in Section 3, seemed to work with the concepts of isomorphic groups and convergent sequences (respectively) strictly syntactically, i.e. solely in terms of the concepts' definitions. When they were asked to prove statements, their approach seemed to be to manipulate the definition in a logically permissible way, or in Ben's words, to 'start off with the definition ... and see what happens from there'. In contrast, the algebraists, in Section 2, and Adam, in section 3, were able to describe instantiations of groups and sequences. Further, they were able to use these as a basis for making what might be called global or intuitive observations about these concepts and translating these into formal reasoning. Using these data, we can now make some general observations about the differences between syntactic and semantic proof productions.

5.1. *Skills and knowledge necessary for syntactic and semantic proof productions*

In this paper, we have reported studies in which students were required to produce proofs that focussed rather tightly on a particular mathematical concept. In Section 3, the undergraduates and doctoral students were asked to write proofs about isomorphic groups. In Section 4, Ben and Adam were asked to construct proofs about convergent sequences. Syntactic and semantic proof productions about these concepts appeared to require the prover to possess qualitatively different understandings of these concepts or, possibly, to develop these understandings during their proof attempts. In this sub-section, we describe what types of knowledge of a concept we believe are necessary for one to produce a syntactic proof and for one to produce a semantic proof.

The abilities and knowledge required to produce syntactic proofs about a concept appear to be relatively modest. The prover would need to be able to recite the definition of a mathematical concept as well as recall important facts and theorems concerning that concept. The prover would also need to be able to derive valid inferences from the concept's definition and associated facts. We say that one who possesses these skills has a *syntactic knowledge* or a *formal understanding* of this concept.

The knowledge required to produce semantic proofs about a concept is considerably more complex. Based on the data presented in this paper, we hypothesise that semantic proof production requires the following abilities on the part of the prover:

- One should be able to instantiate relevant mathematical objects. For instance, the algebraists were able to instantiate groups as group multiplication tables (and in other ways) and Adam appeared to instantiate a convergent sequence as a graph of a sequence that eventually became very close to its limit.
- These instantiations should be rich enough that they suggest inferences that one can draw. Group multiplication tables and graphs of sequences are known to be powerful aids to algebraic and analytical reasoning respectively.
- These instantiations should be accurate reflections of the objects and concepts that they represent. That is to say, these instantiations should not suggest that the associated concepts have properties that are inconsistent with the formal theory.
- One should be able to connect the formal definition of the concept to the instantiations with which they reason. For instance, Adam was able to describe why his instantiations of convergent sequences had the properties that they did in terms of the definition of convergent sequences. Likewise, the algebraists were able to specifically describe how a group isomorphism could be thought of as a mapping that re-labels group multiplication tables. Furthermore, some algebraists were able to explicitly describe how they could translate intuitive observations based on these instantiations into formal mathematical arguments.

We say that one who has the abilities and knowledge described above has a *semantic understanding* or an *effective intuitive understanding* of this concept.

5.2. *Limitations of syntactic proof production*

We do not feel that producing a proof syntactically is necessarily a 'bad' thing to do. On the contrary, we believe that syntactic proof productions can be very valuable in certain situations. For instance, they can be used to prove counter-intuitive results, to build one's intuition about a concept (cf., Pinto and Tall, 1999), to systematise a mathematical theory (deVilliers, 1990), and to verify that the new definition captures the intuitive essence of a mathematical concept (Weber, 2002a).

However, we would argue that having the ability or inclination to only produce syntactic proofs has two significant drawbacks. First, within a particular domain, if one can only produce proofs syntactically, the scope of the statements that can be proven is likely to be rather limited. In the first study reported in section 3, the undergraduates could prove very few of the theorems despite having the factual knowledge and the procedural ability to apply this knowledge required to do so. Second, syntactic proof production can leave the prover unsatisfied with his understanding of the proof. This was illustrated in section 4. Although Ben was capable of proving statements about limits, he nonetheless felt that he lacked a complete understanding of the concept and seemed somewhat unconvinced by his own arguments. Both of these points are discussed in more detail in the following section, in which we relate our above observations to other theories in mathematics education.

6. Discussion

6.1. *Convincing and explaining*

In the mathematics education literature, a distinction is commonly drawn between proofs that convince and proofs that explain (e.g. Davis and Hersh, 1981; Hanna, 1990; Weber, 2002a). A proof that convinces is an argument that establishes the mathematical veracity of a statement. Such proofs are typically highly formal, and their function is to remove all doubt that a statement is true. A proof that explains is an argument that explains, often at an intuitive level, why a result is true.

In our terms, a syntactically produced proof may be convincing to the prover, in the sense that they believe it is logically correct, without being at all explanatory. Indeed, the statement to be proven might not be about anything meaningful and may simply be understood as a string of symbols. A correct semantically produced proof, on the other hand, would involve the prover being able to relate the formal manipulations to the structure of, or transformations of, instantiations of relevant objects. It would thus be both convincing and explanatory. Of course, 'proofs' produced in this way may be both convincing and explanatory to the prover *without* being correct relative to the formal theory–the final section on concept image discusses this point further.

In either case, the combination of conviction and explanation may be related to Fischbein's (1982) distinction between *formal extrinsic conviction* and *internal intuitive conviction*. The former type of conviction is obtained by formal or symbolic argumentation, and thus would be the type obtained by producing a proof syntactically. The latter type of conviction comes from having an internally meaningful intuitive sense of why a statement is true, to the point that the statement appears obvious. We would argue that in a syntactically produced proof it might appear obvious *that* a statement is true, if this can be derived by a simple step from another which is already believed to be true. However, an intuitive sense of *why* a statement is true depends, once again, upon relating this statement to instantiations of objects, and so is related to the type of conceptual understanding necessary for semantic proof production.

6.2. *Instrumental and relational understanding*

Our distinction between syntactic and semantic proof production has an analogy with Skemp's distinction between instrumental and relational understanding in mathematics (Skemp, 1976, 1987). Specifically, one who produces a proof syntactically may be said to understand *what* to do, and one who produces a proof semantically may be said to understand both what to do *and why*. Here, once again, understanding *why* an approach is sensible involves being able to explain the way in which its associated formal manipulations are reflected in the structure of instantiations of objects and/or in transformations of these. With this in mind, we may extend an analogy of Skemp's to illustrate the greater efficacy of semantic proof production.

Skemp compared possessing instrumental and relational understanding, respectively, to having knowledge of a number of fixed routes from place to place within a town and having a map of that town. We do not want to speak about fixed routes, since proof production generally involves more decisions than does following a typical procedure in school mathematics. However, we suggest considering one man who is familiar with traffic laws and has a list of a town's streets, every place two streets intersect, and each street's relevant features (e.g. is each street one way? Can one make a left turn at this intersection?), and a second man who owns a map of the town. Suppose a stranger asked each man for a set of directions for how to get from a point A to a point B. The first man could, at least in principle, complete this task some of the time. At a very crude level, he could piece together random itineraries starting at point A until he stumbled upon one that concluded at point B. Of course, one could imagine more sophisticated strategies that one can use to accomplish this task, but in a town of even a moderately large size, there will be dozens of streets and hundreds of intersections. A relatively unguided search through a list of these streets and intersections would not likely produce a set of directions from A to B. However, if one had a map of the town, one could simply locate A and B on the map and 'see' a path from A to B. Giving directions would consist of translating the path on the map to verbal instructions.

We compare the first man described above to an individual who is producing proofs syntactically and the second to one producing proofs semantically. Just as most streets in a town intersect many other streets, at any given point in a proof, there are many valid inferences that can be drawn that might seem useful to an untrained eye (e.g. Newell and Simon, 1972; Weber, 2001). Hence, writing a proof by syntactic means alone can be a formidable task. However, when writing a proof semantically, one can use instantiations of relevant objects to guide the formal inferences that one draws, just as one could use a map to suggest the directions that they should prescribe. Semantic proof production is therefore likely to lead to correct proofs much more efficiently.

6.3. *Concept definition and concept image*

To conclude this paper, we would like to situate our contribution specifically with respect to the notions of concept image and concept definition. Tall and Vinner (1981) define one's concept image to be one's total cognitive structure concerning that concept. In the mathematics education literature, this term has been used in more or less inclusive ways. However, in many cases, one's image of a concept corresponds closely with our notion of the instantiation(s) an individual uses, especially those that are generalised or prototypical representations of the concept (e.g. Vinner and Dreyfus, 1989). Certainly one's instantiations of mathematical concepts would form a subset of one's concept image. In these terms, one might say that a syntactically produced proof is based solely on the concept definition while a semantically produced proof also makes use of this aspect of one's concept image. Or, in Vinner's words, that a syntactic proof production is a purely formal deduction while a semantic proof production involves deductions based on intuitive thought (Vinner, 1991).

Previous research on concept image has primarily demonstrated three things. First, students' images of mathematical concepts are often inconsistent with the corresponding formal definitions. Second, even when students can accurately state the definition of a concept, an inaccurate concept image may persist and can significantly hinder their formal reasoning (e.g. Tall and Vinner, 1981; Dreyfus and Vinner, 1989; Vinner, 1991). Third, in advanced mathematics courses such as real analysis, students often produce intuitive arguments based on their concept images, and that these do not constitute proofs (e.g. Alcock and Simpson, 2002).

The direction of the results presented in this paper differs from that of these previous studies. The work presented here did not focus on students' invalid arguments, nor on exposing a mismatch between the formal concept definition and students' concept images. Rather, these studies illustrate the way in which working semantically, by using instantiations drawn from one's concept image to guide the manipulation of a concept's definition and the introduction of other relevant properties, can be a significant factor in one's ability to produce correct proofs.

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