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# On the pedagogical insight of mathematicians: 'Interaction' and 'transition from the concrete to the abstract'

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**Abstract**

In this paper we draw on a 16-month study funded by the Learning and Teaching Support Network in the UK and entitled *Mathematicians as Educational Co-Researchers*. The study's aims were two-fold. Primarily we intended to explore mathematicians' reflections on issues identified in the literature as highly topical in the area of teaching and learning of undergraduate mathematics. We also wished to explore the conditions under which mutually effective collaboration between mathematicians and researchers in mathematics education might be achieved. Participants were 20 mathematicians from 6 mathematics departments and the study involved a series of Focus Group Interviews where pre-distributed samples of mathematical problems, typical written student responses, observation protocols, interview transcripts and outlines of relevant bibliography were used to trigger an exploration of pedagogical issues. Here we elaborate the theme 'On the Pedagogical Insights of Mathematicians' as it emerged from the data analysis. We do so in two parts: in the first part we present the participants' reflections on issues of interaction and communication within the context of teaching and learning in higher education. The data suggest that the lecturers believe that mathematical learning is achieved more effectively as an interactive process and recognise that lecturing is not a method generally conducive to interaction. However, they discuss ways in which interaction can be achieved and refer to seminars, tutorials and feedback to students' writing as other opportunities for interaction that must not be missed. In the second part, we focus on the lecturers' pedagogical reflections regarding the abstract nature of university mathematics and, in particular, the ways in which teaching can facilitate the transition from the concrete to the abstract. We conclude with a brief evaluation of the project by the mathematicians themselves.


*Keywords:* Advanced mathematical thinking; Mathematicians as educational co-researchers; Collegiate mathematics education; Lecturing; Mathematics education; Real analysis

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## 1. Introduction

Research into the learning and teaching of mathematics at tertiary level has gained momentum in recent years due partly to the decline in the numbers of students undertaking further studies in mathematics (Hillel, 2001). Moreover, as universities become accountable for their teaching (Alsina, 2001), university mathematics lecturers are becoming more aware of the need to reflect on pedagogical issues. New forums have been created where mathematicians and researchers in mathematics education come together and discuss issues of teaching and learning mathematics. One of those, for example, is the University Mathematics Teaching Conference (http://www.umtc.ac.uk) which runs yearly in the UK. Awareness of the need to engage with mathematics education appears also from within the mathematics community. For example, in the October 2003 issue of the *Notices of the American Mathematical Society* William G. MacCallum explains how research in mathematics education has the status of "feeling more like doing mathematics than like teaching" as it goes through the three stages of Discovery, Publication and Review. After discussing a few examples of work in mathematics education originating within four mathematics departments in the US, he then reflects on the collaboration between mathematicians and researchers in mathematics education:

> Collaborative efforts between mathematicians and mathematics educators are sometimes hampered by a general lack of mutual respect between the two fields. Therefore [*projects should exist*] that are models for how mathematicians and mathematics educators can work together. (p. 1097)

The importance of this quote lies in the fact that it appeared in a *mathematics* journal rather than in a *mathematics education* journal.

Within the community of mathematics education there is wide agreement about the importance of involving mathematicians in educational research (Artigue, 1998). However, often there are difficulties doing so. For example, Sfard (1998) highlights the difference between the research paradigms of the two communities:

> On the one hand, there is the paradigm of mathematics itself where there are simple, unquestionable criteria for distinguishing right from wrong, and correct from false. On the other hand there is the paradigm of social sciences where there is no absolute truth any longer; where the idea of objectivity is replaced with the concept of intersubjectivity, and where the question about correctness is replaced by concern for usefulness. (p. 491)

She also suggests that this is one of the main hurdles that the two communities need to overcome for collaboration to become feasible.

Despite recognition of the importance of collaboration there are not very many examples of it. We were able to find only few examples in recent literature. First, chronologically, is Mura's study (1993) about images of mathematics held by mathematics university lecturers. Second is Burton's study (Burton, 1999a, 1999b; Burton & Morgan, 2000) of the writing of professional mathematicians and their perception of mathematics epistemology. Mathematicians in universities in the UK were interviewed on their views of "how they come to know mathematics and how they know that they have come to know mathematics". Third is the *Undergraduate Mathematics Teaching Project* by Jaworski, Nardi and Hegedus (Jaworski, 2002; Nardi & Jaworski, 2002; Nardi, Jaworski, & Hegedus, 2005). In this study, university mathematics lecturers were regularly observed in their weekly tutorials and then were invited to interviews where they reflected upon critical incidents from the tutorials. This study aimed at the mathematicians' involvement as co-researchers rather than 'subjects of research' (Mason, 1998). More recently the work of Alcock and Weber (Weber, 2004; Weber & Alcock, 2004) has pursued the involvement of mathematicians in generation of mathematics education findings and has contributed to stimulating increased interest in this so far under researched area. The study we report in this paper extends this involvement even further (as we demonstrate in Section 2). The study builds on the tradition of studies on teachers' thinking processes as originally conducted within primary and secondary education (see Nardi et al., 2005, for a brief review of the relevant literature).

## 2. Methodology of the study

Participants were 20 university lecturers from 6 mathematics departments in the UK. The study was conducted in two strands. *The first strand*, where the main body of data was gathered, engaged five mathematicians at the university where the two authors work. The participants were volunteers among the members of the mathematics department who responded to an initial presentation of the project. We will call them Lecturers[1] A, B, C, D and E in the interview excerpts. The research subjects and age of the participants are mixed. Lecturers B and C are applied mathematicians in their early forties. Lecturers A, D and E are pure mathematicians. Lecturer E is in his early forties, Lecturer A is in his early fifties and Lecturer D is in early thirties and just started a lecturing job at this university. They are all white, male and British, apart from Lecturer A, who is not British and was educated in Continental Europe. With this team we held six meetings over six months during one academic year (approximately one meeting per month from November through April). Each meeting lasted approximately four hours and revolved around the discussion of Datasets (see the following) specially prepared for each meeting by the two authors. *The second strand*, aiming at widening the spectrum of participation and at triangulation, consisted of similar discussions of the first five Datasets with groups of university lecturers from volunteering mathematics departments elsewhere in the UK.

### *2.1. The methodological tool*

The methodological tool employed for data collection was Focus Group Interviews (Wilson, 1997). The rationale behind this choice (see also Nardi & Iannone, 2003) was that this would allow us to observe interaction between team members with different mathematical and cultural backgrounds. Moreover, even if the focus of discussion was provided by the researchers in mathematics education (via the choice of the topic of the Dataset) we wished not to provide any interpretation of the data. The aim in this case was to seek involvement in the process from a position of *creating ideas* rather than from a position of *responding to ideas*. As Madriz (2001) explains:

> . . . the researcher usually dominates the whole research process, from the selection of the topic to the choice of the method and the questions asked, to the imposition of her own framework on the research findings. Focus groups minimise the control that the researcher has during the data

[1] In the text we refer to the participants of the study as Lecturers. Meanings of this term differ across different countries. We use it here to denote somebody who is a member of staff in a mathematics department involved in both teaching and research.

> gathering process by decreasing the power of the researcher over the research participants. The collective nature of the group interview empowers the participants and validates their voices and experiences. (p. 838)

The Focus Group Interviews carried out with the team in the first strand were digitally audio recorded and fully transcribed. The interviews carried out in the second strand were also digitally recorded and a protocol (a narrative account of the interview) was produced.

### *2.2. The Datasets*

Each Dataset[2] is a document of about 14 pages given to the participants before the meetings and prepared by the two authors. It contains examples of students' written work or protocols of observation of tutorials (from the authors' previous work: for example, Iannone & Nardi, 2002; Nardi, 1996) and a short list of main points that such extracts might raise. It also includes a brief literature review concerning the particular issue treated, together with a list of relevant references from the mathematics education literature. Each Dataset centred on a key issue regarding the learning of mathematics at university level as explored in the authors' previous work and in the literature on advanced mathematical thinking (e.g. Tall, 1991). This list is by no means exhaustive. The six key issues for the six Datasets were:

1. Formal mathematical reasoning I: students' perception of *proof* and its necessity;
2. Mathematical objects I: the concept of *limit* across mathematical contexts;
3. Mediating mathematical meaning*: symbols and graphs*;
4. Mathematical objects II: the concept of *function* across mathematical contexts;
5. Formal mathematical reasoning II: students' enactment of *proving techniques*;
6. A *Meta-cycle*: collaborative generation of research findings in mathematics education.

### *2.3. Data analysis*

The data analysis of the interviews was carried out in the spirit of Data Grounded Theory (Glaser & Strauss, 1967). Once the recordings had been fully transcribed, the two authors — independently for the purpose of analytical triangulation and validation (Lincoln & Guba 1985) — broke the transcripts into *Episodes*. Each Episode is a self-contained section of the transcript with a particular focus. These Episodes form the analytical units of the data processing. Following a comparative scrutiny of the two authors' Episode breakdowns, a final list of 80 Episodes was agreed. Each Episode was then turned into a *Story*, a narrative account in which the authors summarise the content of the Episode, while focusing on particular aspects of the Episode by occasionally quoting verbatim from the transcript. (See Appendix A for a demonstration of the process of transforming a transcript excerpt to an *Episode* and then a *Story*.) The Stories, out of the Episodes, were again produced by the two authors independently and then scrutinised collaboratively. Once an agreement had been reached regarding main focal points of each, the 80 Stories were then grouped into five categories as shown in Table 1.

In this paper we focus on the second category.

[2] For an example of a Dataset see Sangwin, Cooker, Hamdan, Iannone, and Nardi (2003).

Table 1

| Category | Number (total: 80) | Content |
|---|---|---|
| 1. On students' attempts to adopt the "genre speech" of university mathematics. | 25 | Reflections on the learning of mathematics at university level (on students' thinking as evident in their mathematical writing and speaking) (Nardi & Iannone, 2005, in press; Nardi, Iannone, and Sangwin, 2004, submitted) |
| 2. On the pedagogical insight of mathematicians: teaching as initiation into the "genre speech" of university mathematics. | 25 | Reflections on the teaching of mathematics at university level (perceptions of one's role as a university teacher and on relevant pedagogical practices) |
| 3. On the impact of school mathematics on students' skills, perceptions and attitudes. | 4 | Nardi et al. (2003) |
| 4. On the mathematicians' own mathematical thinking and the culture of professional mathematics. | 20 | Reflections on the ways mathematicians do research in mathematics and on their own experience as learners (Iannone, manuscript in preparation). |
| 5. On the relation, and its potential, between researchers in mathematics education and mathematicians. | 6 | Nardi and Iannone (2004). |

## 3. Explanation of Category 2 and data analysis

### 3.1. *What do we mean by 'on the pedagogical insight of mathematicians: teaching as initiation into the "genre speech" of university mathematics'?*

In this category we have grouped together stories where lecturers investigate their own pedagogical practice, propose changes and improvement to it, all for the purpose of introducing students to the form and content of university mathematics. Our understanding of 'genre speech' is that of Bakhtin (1986), explored further by Van Oers (2002).

> The genre is primarily a social tool of a sign community for organising a discourse in advance and often even unwittingly. It is a style of speaking embodied in a community's cultural inheritance, which is passed to members of that community in the same way as grammar is passed on. A genre is not so much a strict and fixed social norm, but it is a generic system of changing variants and possible utterances that fit into a community's practice; it is some kind of arena or forge where new variants of utterances are created and valued, that contributes to the essential polyphony and dissonances of meaning and discourse. (p. 69)

Let us stress that the focus of the participants' reflections presented here is on more effective teaching *within* the existing lecture/seminar university framework. As we will demonstrate in the data and analysis that follow, the mathematicians may find themselves largely disagreeing with this structure, but nevertheless reflecting on how to teach within it. Moreover, even if there is acknowledgement of the fact that the curriculum may need to change with the changing needs of the students, this is not the main focus of discussion in the interviews. The main focus is teaching practice in all its aspects, with the existing curriculum and in the existing framework as given.

Table 2

| Sub-category | Content | Frequency |
|---|---|---|
| 1. Learning through interaction | On the necessity of interaction between lecturers and students, and amongst the students themselves | 4 |
| 2. Lecturing | Shortcomings of lecturing, and techniques suggested/employed by the lecturers for improvement | 14 |
| 2.1. The clash between different courses in the same year | How different messages about learning mathematics given in different courses are confusing for the students | 2 |
| 3. Seminars and tutorials | What seminars and tutorials aim to achieve, optimal use, timing and format issues | 4 |
| 4. Homework | | |
| 4.1. Choosing homework questions | On selecting and formulating homework questions | 8 |
| 4.2. Use of the homework I | On question setters' intentions with regard to covering mathematical content and addressing students' learning needs | 2 |
| 4.3. Marking the homework | On marking and written feedback to students | 1 |
| 4.4. Producing model solutions | On the use and optimal format of model solutions; the advantage of tailoring them to the responses of the students | 1 |
| 4.5. Use of the homework II | On the link between homework questions and exam questions | 1 |

### *3.2. Data analysis within Category 2*

Following agreement by the two researchers on the Story breakdown, the first author suggested a further breakdown of the 25 *Stories* within Category 2. The breakdown, into eight sub-categories that were interrelated and at times overlapping was agreed upon: a Story could belong to more than one sub-category (which explains why the frequencies in Table 2 add up to more than 25, the number of Stories in this Category). The first four subcategories (see Table 2) revolve around *learning through interaction*, an issue that, while raised by the participants within the context of their experience as teachers of mathematics at university level, is also more generally germane to teaching within higher education. The remaining four sub-categories (see Table 3) revolve around an issue that is more specifically germane to the

Table 3

| Category | Explanation | Frequency |
|---|---|---|
| Representation of abstract mathematical ideas in concrete terms | | |
| 5. The role of examples and numerical evidence | On the role of numerical examples in different areas of mathematics | 5 |
| 6. The use of IT | On the use of graphic calculators and specialist mathematics software | 2 |
| 7. The language of mathematics | The tensions between the formal and informal, symbolic and verbal language in teaching mathematics | 5 |
| 8. Metaphors | Metaphors used by the mathematicians to describe students' learning | 1 |

teaching of *mathematics* at university level, the *representation of abstract mathematical ideas in concrete terms*.

In the following sections we concentrate on sub-categories 1–5, touch briefly on 6, and leave 7 and 8 for discussion elsewhere. We conclude with a brief section on the participants' reflections on how participation in the project has initiated reconsideration of teaching practices (we elaborate these in Nardi & Iannone, 2004).

*3.2.1. Learning through interaction*

The following extract exemplifies the participants' views on the issue of learning through interaction, an issue they would return to repeatedly throughout the period of data collection.[3]

A: Because I actually think that the element of difficulty in solutions is not something that you can pin down on a piece of paper. An argument, between the student and the teacher, ideally, why is this so and by the time I distil it in the model solutions the argument is dead, it becomes part of script. So the element of discourse is [. . .] what the thing is about. And we should have oral examination, and we should have lots and lots of oral interaction with the students to see that mathematics is about arguments. [. . .] *[Lecturer A refers to a 'small table' arrangement in the department where students are invited to congregate and work together]* Maybe there is also an important point into this. Where our students learn the best is in these interactive environments and things like these that they sit around and talk and talk all day. It appears to me that they learn more there than in many of our lectures.

EN asks whether it is possible to incorporate this type of arrangement somewhat more formally into the learning experience of the students.

E: No, really, and it is not easy to follow actually because these things work at a social level and then they work at an intellectual level.

C: And as soon as a faculty member is there, you know, it will distract the interaction.

A: It is so strongly supported that we could not do more, in a way. We encourage them to be there, we give them this space, we say how wonderful it is. They see themselves how wonderful it is, there is no doubt about this, and they know for themselves that it is the thing to do. And this is about as much as you can do. You can't make it obligatory.

E: No . . . And already we have some slight rumblings of difficulty in the university about the fact that we will never use the word0(onderful)- 1 nrning with [

have this happening if it wasn't connected with homework. There need to be some work that they have to do in order to trigger them to do this. [. . .] I think it is a value system driven by departments with different methods of assessment.

The above dialogue is initiated by Lecturer A, who reflects on the absence of communication with the students. The conversation, however, shifts towards the participants' epistemology of mathematical learning (or the *coming to know mathematics* as in Burton, 1999): learning mathematics is better achieved as a discursive endeavour. The participants then talk about one way they are trying to achieve this in their department. The "interactive environments" Lecturer A refers to, are tables scattered around the corridors of the mathematics department and available for students to use for group work. In the words of Lecturer E, mathematics is perhaps epistemologically different from other subjects, and this should be taken into account in assessing learning. This difference at times clashes with rigid university rules where lecturers are required to asses students' work *per individual* therefore making the encouragement of collective students' work very problematic. The collective work of two or three students is not seen as copying, but it is seen as the best learning tool there is, perhaps even better than lecturing. Lecturer B further elaborates the belief in the value of interaction in mathematics (see Mason, 2002, p. 6 on the same issue):

B: Yes, I think that there are several levels of language at work and students talking to each other is such a one level. At an other level there is when the lecturer giving a class, maybe one to one to the student who has come to the lecturer saying:"I don't understand this question". And there is a third level of language that is a language that normally arrives first in questions like this one here and in texts books as well. Now I sympathize with students that have troubles with this very concise language when rather formal lecturers who sometimes speak to them in a language that is appropriate to a lecture hall and not to one to one advice. It is not been informal enough. And something is happening down among students in our undergraduate reading room where they are talking to each other and there is a kind . . . I don't know what it is, some vibe that they are able to tune in to each other to get to each appreciate what is going on.

The reflections offered above by Lecturer B touch upon a theme that runs through the whole of data collection: the varying roles of the language of mathematics. For him, there seem to be three languages at work simultaneously, and each one of them fulfils a different need. First there is the formal language of textbooks and homework questions, which makes the building of mathematics formally and logically sound. Secondly, there is the language of lectures that keeps some of its formal characteristics and is aimed at exposing the material to students. Thirdly there is the informal language used by the students between themselves, but also by the lecturers on a one-to-one conversation with the students, which appeals to the individual intuition and should facilitate understanding of formal mathematical concepts. This difference in *language modes* seems to be connected with the need of conveying concept definitions (see Tall & Vinner, 1981) via the formal language of textbooks, and the need of creating suitable concept images, via the more informal mode of language of a one-to-one interaction between student and lecturer. As for the even more informal mode of language used in interaction between students, this serves the aim of comparing and refining concept images of mathematical concepts. What Lecturer B seems to assert in the above extract is that all the three modes of language are needed and that the aims they serve cannot necessarily be interchanged.

The views expressed here are not new. The debate in the education community on the strength of group work as a learning tool in schools across a number of disciplines has gone on for quite a while. For an example regarding learning mathematics in schools in the UK see the development of the *Cognitive Acceleration in Mathematical Education* scheme by researchers at King's College, London, for teaching mathematics at Key Stage 3 (for further information see www.kcl.ac.uk/kings_college/depsta/education/teaching/CAME.html). What is interesting to observe is that the acknowledgement of importance of interaction by these mathematicians (both intellectual and social, as clearly stated by Lecturer E and as propagated by a number of writers in, for example (Holton, 2001) with regard to the teaching of mathematics at university level) clashes with what studies have reported to be the students' experiences of learning mathematics across the educational levels. For example, in Almeida (2000), university students describe mathematics as an isolating subject, as opposite to other subjects like English. In addition, in Nardi and Steward (2003), disaffected 13/14-year-old students describe their longing for learning environments where doing mathematics is not a solitary task but where collaboration with peers is actively encouraged.

### *3.2.2. Lecturing*

There was wide agreement among participants that lecturing is not the format that best suits the teaching of mathematics. Lecturer C exemplified this main shortcoming as follows: during a lecture often students seem to "be nodding in the right points", the lecturer then thinks that the point she/he tries to make is "blindingly obvious" and moves on. When the students' written work arrives, it is "gibberish"! As Lecturer C puts it: in the context of the lecture it is actually "quite hard to judge what students find difficult". Despite their reservations towards the effectiveness of lecturing, the participants seemed to take this format of exposition of material as given and did not discuss alternatives. Instead they discussed in some detail ways to optimise its potential. Below, we exemplify their discussions in terms of *Tactics* to improve learning through lecturing and *Desired Outcomes* of lecturing.

*3.2.2.1. Tactics.* We use the term *tactics* for improving learning through lecturing in Mason's (2002) sense, strategies for achieving short-term goals. In the following we offer two examples of tactics suggested by the participants.

*Using stepping stones to facilitate the transition from less to more complex ideas*. This way of proceeding resonates with Hazzan's (1999) theory of reducing the level of abstraction. In the example that follows, the participants discuss it in the context of Analysis and in particular the convergence of sequences and series (see Appendix A for the text of the question). Lecturer E, who lectured the Analysis course, describes his tactic to start with a series where the geometric sum is known: "I aspire to have odd moments in first year analysis where they [*the students*] meet with something they know". In his view students will benefit from encountering something they already know when starting something that is very unfamiliar to them, as, for instance, convergence of a sequence. Another of the participants, Lecturer B, is skeptical about this approach as it may be seen as applicable only in the context of this particular exercise. Lecturer E agrees, and remarks that this approach can only last for a limited time as "immediately afterwards you part company with this sort of approach, this way of taking a series that you can compute what the sum is". The idea is that the stepping stone has two functions: a *cognitive* one which bridges the gap between calculating a sum and determining a limit; and an *affective* one that serves as a confidence-building exercise in a topic seen as one of the most difficult in the first year of a course in mathematics (e.g. Tall, 1991; Tall & Vinner, 1981).

The observations of Lecturer A in the context of Group Theory are very similar. In this case the stepping stone is represented by group tables introduced at the beginning of the course, just after the formal definition of a group.[4] This group table approach was introduced by the lecturer as a facilitator concept: students cherish the idea of something which is “doable” in the midst of a subject that is very new and abstract, and it is also only used at the beginning of the course. Of course, this approach is not free of risks, the participants observed. The use of group-tables in Group Theory could give the false impression to students that each table corresponds to a group (which is not the case for groups of order greater or equal to 6) and also falsely encourage let them overlook other properties of a group such as associativity (Iannone & Nardi, 2002). The lecturers demonstrated an awareness of these risks and acknowledged that it is difficult to achieve a balance between wanting to support students’ learning by resorting to familiar ideas and wanting them to perform the leap towards a new abstract idea.

*Spell out for students the different roles of definitions and theorems and devise examples in lectures which highlight logical reasoning skills*.[5] In the participants’ view, students arriving at university lack basic logical reasoning skills (Nardi, Iannone, & Cooker, 2003). Therefore, to successfully prove and use mathematical theorems they need to learn to use definitions. In the context of an exercise involving the Limit Comparison Test for convergence of series, Lecturer E suggests that “what I try them [*the students*] to do is to use the definition to guide them through the proof. So I can say the definition and where there is ‘for all’ in my solution I say ‘given’ and where it says ‘there exists’ I must do a calculation and construction so on. So the order is completely crucial”. Therefore, lecturers seem to recognize the need to improve students’ logical reasoning skills. On the same subject, again Lecturer E: “A very, very similar example [*to the one looked at*] is gone through in the lectures in great detail in order to try just to get them to unpack meaning into . . . to understand how do you witness a theorem’s falsity”. To some extent this debate on logical reasoning connects naturally with the widely debated issue of proof in mathematics.[6] Proof is one of the most highly topical issues on which the mathematicians reflected, but a description of the findings concerning this is outside the aims of this paper.

Also connected to the issue of logical reasoning is the perceived clash of instructions given by different courses, namely in pure and applied mathematics, in the same year. In the words of Lecturer E: “I mean I teach them in the spring the derivative, and the measure of the slope, and this is the semester after they were solving differential equations” Therefore, “It is very difficult to motivate these quite difficult, very rigorous proofs of what the derivative of $\sin(x)$ is, and that requires the definition of $\sin(x)$ and so on.” This issue of inter-university conflict is also extensively discussed in Nardi (1999) along side with school-university conflict and intra-university conflict.

*3.2.2.2. Learning outcomes.* While the tactics proposed by the participants generally aimed at achieving short-term goals, they often also discussed more long-term desired outcomes regarding the learning of their students. Among these is the acquisition of *skills of synthesis* as Lecturer E discusses in the context of an exercise involving the Limit Comparison Test:

E: Various things have been attempted. John [*another lecturer at the same university*] once got them [*the students*] to write down, write out all the theorems, just statements of theorems in the Number Theory

[4] For the first appearance of group tables in a Group Theory text see Burnside (1911).
[5] See, for example, Barnard (1995) with reference to negation of logical statements.
[6] See, for example, Schoenfeld (1994) to mention just one work concerned with proof and also with teaching and curriculum change.

course. It takes two sides of A4. And . . . some accepted that. One very interesting technique that I have seen done is to get them to write down anything they like from the course, that fitted on two sides of A4, hand it in, in advance, and then it is on their desk in the exam. And they choose what to write down and they do it in advance. Fantastic thing! It makes them think through the course and think what matters and think what they need to know.

This desired outcome connects directly with the second tactic we have described above. It is important, in the eyes of the lecturers, that students understand the most relevant parts of a course, as this will guide them through learning the strategies needed when writing a proof, or solving a problem. Hence the implementation of a tactic that will fulfil this need.

*Sense-making through a repertoire of examples*. A lecture course, the participants frequently suggested, should contain a robust set of examples to help students make sense of what they hear. In the context of analysis, the mathematicians wish for the students to have a *toolbox of examples*, such as basic analytic functions, which they can retrieve when needed. For more evidence of the importance of this see Nardi, Jaworski, and Hegedus (2005). Within the discussion on the use of examples, the participants elaborated how students should learn how to handle *limit-examples*. What they seem to mean here is that students should learn how to verify the validity of a proposition by considering what it implies in a limit-condition. To exemplify this we refer to the words of Lecturer E in the context of an exercise in Group Theory:

E: The sort of thing you wish for is that they would have this robust set of examples in their minds somehow so when they read something they can think: well . . . M could be a trivial group and G could be anything. And in the light of that possibility everything I say should make sense. That is what we really should wish for. [. . .] I think that very few of them do.

We note here how this issue naturally connects with the development of logical reasoning skills discussed in Section 3.2.2.1. It is only via a correct logical reasoning and a good understanding of the basic properties of all the definitions and theorems involved in proving a proposition that the students can arrive at viewing this in the limit-case, as advocated in the extract above.

### *3.2.3. Seminars and tutorials*

The main point of the discussion on seminars and tutorials is that this is often the only opportunity for interaction between lecturers and students. As the students are accustomed to the more passive role of attending lectures, the lecturer needs to invent ways in which the students become more involved in the seminar/tutorial process. One such way, suggested Lecturer E, is to ask the students to read a statement they have written down aloud in the class. In the course of doing so, he claims, students often realise whether their statement is correct or not. The size of the class is a crucial factor in the success of this, he continues: small tutorial classes are more appropriate than large problem solving seminar classes. As the Datasets discussed in the interviews often contained transcript extracts from observed tutorials in an UK university where one to one or two to one tutorials are the norm, unlike the typical 6+ size of classes in most UK universities, class size was discussed extensively. Maybe surprisingly, the mathematicians think that seminar classes of 15–20 students might be more effective than very small tutorial groups. The reason is that the former, while allowing a reasonably intimate exploration of each individual student's ideas, does not become overly uncomfortable for students who would have to expose their ideas and doubts in front of their peers as it might be in the latter.

Beyond class size, orchestrating the interaction in ways that builds the students' confidence is of paramount importance. According to Lecturer B, for example, it is crucial to first hand out exercises which will not be assessed. These can be discussed in the seminar "to build up" the students' confidence. The homework, which will be assessed, can then contain similar exercises to the ones seen in the seminars. In his opinion this tactic helps the students not to feel intimidated by material they are not familiar with and enhances the likelihood of the student actually engaging constructively with the tutorial process. In the extract below Lecturers B and A highlight how the students are often less willing to engage in the process, as they may not realise the potential learning benefits lying within such participation.

B: You might catch a student in the act [*of learning*] in the seminar. One says, well, I make an appropriate comment . . .

A: Yes, there are various stages towards moving towards where they should be but whether if they actually make a sufficient effort to reach that point so they [*the students*] probably say no . . . there are other obligations and we actually don't care.

As EN observes that part of a successful orchestration of the interactive process is to avoid the tutorial becoming a 'mini lecture' and ensuring that the invitation to participate is absolutely clear to the students, another element of successful orchestration is highlighted by Lecturer A:

A: Yes, you see, what we observe here is actually a degree of detachment from what should happen and what the teacher wants you . . . intends the student, you know, to reach . . . [. . .] and where the students really is. And in most of these examples an additional nine days would have made it obvious that of course this couldn't have happened. So the way things are structured here is not in a way . . . we don't have the time to so . . . so there are out of sink. [. . .] But you see the seminar . . . the problem sheet is written before you give the class, you write this in September and then you try to be there and then it should happen but these things aren't exactly.

Between planning and conducting a seminar or a tutorial often lies a period of weeks or even months. The problem sheet is planned in advance, therefore without much tailoring to the needs of the particular r358Sor358So Lecm58So-24conduct8So semi8Sotherm58Sor358So 8So s,thersens4t.36.1(ity0(theseo0(u)0(t)-er) s'eper git82.3t42.3 T2.3.

distributed after marking (see Appendix A for an example of these notes). In the following we outline the participants' views on these issues.

*3.2.4.1. The choice of questions to appear in the homework.* One of the primary reasons for including a question in a problem sheet for Year 1 students, the cohort that most of our participants had teaching responsibilities for, is: in what ways does this particular question foster the skill of mathematical writing that Year 1 students need to acquire? By this the participants mean both writing using the mathematical symbols correctly as well as writing in full sentences and with explanations. One such question, for example, required that students translate the symbolic statement of convergence of a series into a verbal statement and write its negation. Lecturer E (who set the question) explains the rationale behind it:

E: They have to get used to it. To negate the statement... you can algorithmically negate changing there exist with for all and so on ... And then you come up with a very weird statement, not the sort of statement you can actually reason with effectively. You need to modify it by resort to its actual meaning. So the classic one in analysis is to say [...] there is $n$ such that for all $n$ bigger than $N$ this implies that, let's say. And how do you exactly negate that? Well, if you negate it formulaically you end up with the kind of statement that is correct but it is not useful and it is not what you are going to use. You have to think: what does this mean now? And then write it down in words there are arbitrarily large such and such where this is smaller than that or bigger than that, or whatever. And ... there is just no way around it. You have to be fluent in both facets of the language of the underlying meaning, and that is just hard. And the idea that one could do this entirely symbolically or entirely not symbolic I think is just mistaken. The meaning has to be in their heads and they have to have the ability to write it in words and in symbols and jump between them. And I don't see it very widely, I wish there was, to be honest. It would be marvelous if they all did it.
A: And when students are fluent ... which of the two languages do you think, let me just say, when they are more confident, towards which of the languages would they gravitate to?
E: I don't know ... [...] they have to use both.
A: Yes, but then which one would they then use because they are more at ease with it?
E: I don't know ... A mixture, they have to use a mixture. You just can't let ... even at this time they can't write their mathematics purely verbally or purely symbolically, they just can't. And we can't.
B: We are trying to persuade the students of something, are we? We are trying to persuade them that not all formulas are valid under all circumstances. Writing down the elements into a rather restrictive set in order to get the statement to be true is important. In a sense we are trying to get them to use the symbols in order to make sure that they are specifying the sentence in an unambiguous way.

In the context of using mathematical symbols the participants observe lack of precision and confusion of meaning. Again Lecturer E refers to one of the desired outcomes of lecturing: students need to learn to reason logically and need to learn to do so both using symbolic language and verbal language (Laborde, 1990). But above all they need to have acquired possession of the "actual meaning" of the statement they are trying to express. In the above extract, mathematicians are referring to symbolic language (of the type of definition of convergence, say) rather than to the symbols which represent process and concepts (Gray & Tall, 1994) as, for example, an integral or $f(x)$ for a function of $x$. Therefore, their observations are more in the realm of *literacy in the language of mathematics* (Bullock, 1994). The choice of speech-mode (verbal or symbolic) to adopt is crucial for the successful solution of a mathematical problem. Interplay between these two modes is not replaceable by using one mode only, as again Lecturer E asserts in the extract above. Finally, Lecturer B offers the rationale behind it: students have to learn the rules of logic as these regulate the use of symbolic language. But they also need to learn how symbolic language helps to make unambiguous statements. It is interesting to observe how this connects with the remark by Lecturer B reported in Section 3.2.1. There he considered the three levels of languages which are needed to teach and do mathematics; here he reflects on the two facets of the formal language: the verbal and the symbolic. We say *formal language*, because it appeared from the interview that the mathematicians were talking about *formal verbal language* of the type of, for example, replacing the universal quantifiers in a definition of convergence with the words "eventually" and "arbitrarily" in a formally determined way. Here Lecturer B is making clear the need for a teaching agenda to stress the importance of formal language to students: formal symbolic language is needed to avoid ambiguity in meaning.

*3.2.4.2. The purpose of homework.* As highlighted in Lecturer A's 'detachment' comments in the Seminar/Tutorial section above, it is crucial that teaching is tailored to the needs of the students. Doing so involves primarily understanding what these needs are and a scrutiny of the students' regularly submitted written work is a major resource for this task. Lecturer E elaborates on this role of the homework and contrasts it with its more conventional perception as an assessment exercise:

E: I think there is a huge problem with the idea that ... these sheets are, you know, I never understood why they used them to ... attain marks where in fact we want to use them to teach mathematics. These are two completely different things. We use them to obtain marks, to test knowledge and to teach mathematics. [...] To me it is sort of absurd, when I think what the intent behind a question like this is, it is absurd that this student gets two out of ten and this student gets seven out of ten and that is carried forward to their marks in the course. That is absurd. The three out of ten person has done a fantastic job, they observe that they don't understand this so we have an opportunity to teach the maths, and maybe they should get the seven!

Therefore, for Lecturer E there is an in-built contradiction in the function of homework[7] as simultaneously a tool for formative (identifying students' needs and modifying one's teaching accordingly) and normative (assessing students' progress) assessment. In his view, the presence of questions or an incorrect statement in the homework are in fact the most productive ways of understanding students' needs. A student who introduces a question is to be rewarded as she/he has given the lecturer the chance to "very efficiently teach". Here again we notice the conflict between strict university rules and the epistemology of mathematics as perceived by the practitioners. The mathematicians feel that a mistake in the solution of an exercise allows them to respond to students' learning needs and should not be penalised, as the homework is often the only way to communicate with students.

*3.2.4.3. Marking the homework and producing notes on solutions.* The participants see marking the homework as another opportunity for establishing a dialogue with students; providing detailed written feedback, a tactic also strongly advocated by Mason (2002), is to them the "second best" thing to face-to-face interaction with the student. Students also seem to appreciate an elaborate response — and, as Lecturer A puts it in the extract below, a response in the seminars that reflects the lecturers' understanding of the students' "collective errors":

[7] Identified in the relevant literature such as William and Black (1996).

A: In the third year class I have done something that is maybe connected to what you want to ... investigate. I mean, I found out that the types of errors are often rather different from what I think they would be. So as I haven't written the model solutions or suggested solutions, while in the past I would have done this before I look at solutions. Now I do it afterwards. And now I see that the students out of the sudden realise that [...] in some way I am responding to their collective errors, and they rather like this.
E: And they appreciated it.
A: You see because it is not that I am just saying ... it is not a question of a model solution ... it is a response to what they have done. [...] And it worked quite well. Is not a lot more work.
E: I can remember that as an undergraduate. The solution sheet back that said "I am surprised to find that most of you found this question quite difficult". And it was a rather nice thought, that you were actually ... in a class of one hundred or something, you were actually in dialogue with the lecturer.
B: And students learn that other students have had the same difficulties.

The participants — see also their comments in the Seminar section and previous Homework sections — are adamant about the value of tailoring the presentation of notes on solutions, written by the lecturers and handed out to the students after their homework has been marked, to the students' needs. They back this belief with the positive response by students, and with their recollections of their own experience as students, where this dialogue based on catering for individual student needs was highly appreciated.

### *3.3. Representation of abstract mathematical ideas in concrete terms*

Induction into the genre speech of university mathematics (Bakhtin, 1986) involves a transition from concrete (often originating in the students' school mathematical experiences) to abstract mathematical ideas. In resonance with extensive literature in the field (e.g. Tall, 1991), the participants in this study referred to this transition in a variety of contexts and as an issue that their teaching needs to address constantly. Here we present briefly two of the strategies proposed by the participants to facilitate this transition: numerical experiments and educational technology.

#### *3.3.1. The role of numerical experiments*

The participants' perceptions of the role numerical experiments can play in the process of understanding new concepts or solving problems or determining the truth or falsity of a proposition varied according to mathematical specialism. In the extract below Lecturer E discusses the roles of numerical experiments in the context of Analysis and in particular with reference to identifying the limit of a sequence:

B: Sure. I mean ... In some sense one is very, very constrained by ... just the subject, the nature of the subject [*analysis*]. But I try ... I mean one thing that I try and get them to do ... which again they are very resistant to, ironically, is to do numerical experiments. What is a reasonable way to guess the limit of a mysterious sequence? Evaluate it when n is very large. This is not a proof but it is a very reasonable thing to do. They will not do it and this year I have forced them to do something on Sterling's Formula numerically, in a situation where they have no ... It doesn't matter how much A level maths they know they have no tools at their disposal except numerical and they have to use numerical deliberately.
B: Can I follow that up? Three students that must have been on that course for which you set that work came into my office out of the blue and showed me the question sheet and I saw what they needed

to do. It was an exploratory type question and I said to them that the way that I thought they should approach this is to go and explore. Take a table of values, I think it was $n!/n$, see how $n!$ grows and compare it with the thing that has been suggested in the question. Hence they suddenly realized . . . I think that they suddenly realized that this was an open-ended thing. And then as usual they think they would be expected to using the symbols in the question to somehow conjure up an answer.

E: But that too is something that they have great difficulty with. When they see little $n$ . . . this is potentially a number, an actual number. That is not obvious to them.

So numerical experiments are proposed by the lecturer as a way students can use to get “the feeling” for the behavior of limits. The interesting observation here is that, in the experience of these lecturers, students resist this approach, and this remark might potentially open up the debate about the use of open-ended questions in teaching mathematics in school. It seems to be the case that students resist this approach, as they are not used to it. In the words of Lecturer B students arrive at university believing that solving a mathematics question consists in rearranging “the symbols in the question and conjure up an answer”. Therefore, getting them used to look at open-ended problems can be very productive to develop their skills of intuition.

Not all our participants agree upon the usefulness of numerical experiments. Lecturer A, an algebraist, has his reservations as to whether their value extends beyond Analysis or maybe Physics: Lecturer E, an analyst, disagrees and says that even in research, mathematicians use examples, “This is what I do, anyway.” Lecturer A agrees, but “only if it creates structure insight”. He then offers his experience “as an algebraist”: computations only help verify parts of a conjecture. Lecturer E now stresses that there are things that can only be approached with computations in a first instance, like Sterling’s Formula, and using a tool like MAPLE[8] can help create insight. Lecturer B, an applied mathematician, agrees with this and suggests that using these tools might prompt the students to search for a rigorous explanation. Lecturer D, another analyst, adds that the danger of this is that the students might think computations are all they need. Lecturer A is still not convinced. “Personally”, he says, “I do not want to see the asymptotic behavior of $n!$ even if I deal with it all my life!”. The tensions evident in this extract of conversation were typical across the body of collected data and were particularly intensified when the participants were discussing the role of visual means (e.g. graphs, tables, diagrams, etc.) in learning mathematics. We return to these tensions in the context of examining particular examples of student writing elsewhere (for updated information see http://www.uea.ac.uk/∼m011), but we offer here a glimpse into this wide and fundamental issue with reference to the participants’ views on the use of educational technology as a facilitator of the transition from concrete to abstract mathematical ideas. Of course the use of IT is a well-researched and topical area that we touch on only briefly here.

### *3.3.2. The use of technology in first-year mathematics*

The participants referred to educational technology mostly in terms of graphic calculators and more advanced mathematical software. Often resorting to technology is the only way to gain mathematical insight. In the example below, the students were invited to gain insight into Sterling’s Formula by using the software MAPLE.

[8] MAPLE is a registered trademark of Waterloo Maple Inc, details of this computer algebra system may be found at http://www.maplesoft.com.

E: Ah, and that is a different type of thing. I have seen these sort of things [*graphing calculators*] and that is very different to me. That somehow one needs a mechanical device to convince you that one over root n is small when n is large . . . This problem [*on Sterling's Formula*] was deliberately designed for things . . . where you . . . you have to have no intuition about the size of this thing. That . . . Without MAPLE they can't do this. There is nothing they can do on paper that will help them. And then I think the machine has a very valuable role. I mean . . . That is what computers are for. No to let them do analysis after a lobotomy. Is to let them explore expressions which you cannot . . . you cannot deal with by hand.

B: And it belongs to the experience of the students seeing some very nice convergence . . . that will increase their mental store. Ask them to draw mental graphs that has some intuition I would hope it would be extended . . . Was that part of what you were trying to do when you set that question?

E: Well, I wanted to just encourage them to see that [. . .] you can compile evidence that goes into informing your judgement and . . . And of course historically, in empiricism, [. . .] this is exactly what was done. This poor chap, Sterling, had a great deal of numerical evidence towards . . . what he ends up with.

The use of mathematics software is seen as very productive for the students. This will help them in "building up their mental store" (in the words of Lecturer B). Again suspicion towards educational technology, for example, in the form of graphic calculators, came from participants who seem to prefer leaving things to imagination or the eye of the mind: as Lecturer A says, "If I have the behaviour of a function in my head, then there is no use for the graphing calculator". His concern, in resonance with Lecturer D's reservation in Section 3.3.1, is that these numerical investigations might create confusion between proof and evidence. These tools should help "create insight" but he doubts they do, and the students should be better than the calculator: "I must be actually better than the calculation, I need to prove why the outcome is correct". Overall, however, the majority of participants seemed to agree that technology, appropriated away from "doing Analysis after a lobotomy", can certainly be used for the construction of valuable mathematical insights.

Diversity of views, often amounting to downright disagreement, among the participants — as evident, for example, in the last two sections — was striking across the whole body of collected data. As this diversity often became the trigger for heated debate, the conversation became a very vivid showcase of the participants' views, which is what the data collection of the study primarily aimed at. These debates became possible because of the Focus Group Interview design of the study as well as the capacity of the examples in the Datasets to fuel debate and allow controversial opinions to surface. Apart from the design providing substantial access[9] to the participants' views, attitudes, beliefs, etc., it seems to have fulfilled other functions too. In the concluding sections of the paper we turn to the participants' evaluative comments on participation in the project in order to address what might be seen as the substantive contribution of the study. For most of them, this was their first extensive collaboration with researchers in mathematics education, and towards the end of the study (Cycle 6 of data collection), the discussion revolved around the fragile, yet crucial relationship between mathematicians and mathematics education researchers. Elsewhere (Nardi & Iannone, 2004) we describe their views in terms of obstacles and desired characteristics of this relationship, and of potential benefits for mathematicians who engage as educational co-researchers. Here, we exemplify these views by briefly outlining participants' statements with regard

[9] We consider the design to be part of a methodological contribution of this study.

to the impact that participation in this study had on the raising of their pedagogical awareness, both in terms of the ability to articulate pedagogical reflection (Section 4) and in terms of pedagogical practice (Section 5 where we also summarise the participants pedagogical insights as exemplified in Section 3.2).

## 4. Mathematicians as educational co-researchers: an opportunity for articulating pedagogical reflection — an example

In the Dataset for Cycle 6 the participants were offered transcript extracts from their own interviews from Cycles 1–5 for comment.[10] The encounter with their own "spoken word" prompted the following comments by Lecturer B. In the world of mathematics, he says, it is only the written word that has high status, whereas the spoken word is "kind of transitory". The experience has made him more sensitive towards the "spoken word" he offers in his teaching:

B: And another thing . . . I have become much more conscious about the spoken word. What I can say can have an impact, saying the right thing at the right time when you get one opportunity to introduce the students for the first time to how mathematics works and not fluff the line. That I think has made a big influence on the way I lecture. [. . .] Well, it has a little bit to do with being a bit . . . introducing silences into my lectures while I am writing on the board, [. . .] to help build up to a sentence.

C: It is something that we have talked about. We both go to extremes of lecturing. I don't allow any silence at all; I just gibber while I am writing things on the board while Lecturer B takes long conscious silence and comes up with a gem of a sentence. But these are two extremes.

Therefore, our participants, through facing their own spoken words transcribed on paper, reflect on the significance of their own spoken word in teaching. In this sense, one aspect of the participation in the research process has become an opportunity for reflection on their own practice.

## 5. Mathematicians as educational co-researchers: an opportunity for reconsidering pedagogical practice — an example

The participants appreciated the opportunity offered by the study for reflection upon pedagogical practice and repeatedly stated that such opportunities should be more readily available to university mathematics teachers (e.g. through appropriately content-specific training courses for new and in-service lecturers). They also recalled with appreciation past but disperse experiences of implementing teaching innovations and expressed their concern that the two communities, of mathematics and researchers in mathematics education, have not built sufficiently robust and mutually respectful channels of communication that allow research findings to inform practice (e.g. Alagia, 2003; Mason, 2002; Sfard, 1998). Building on the goodwill, enthusiasm and trust generated during this and our previous studies, we are currently planning a study which will engage mathematicians in a collaborative consideration, implementation and evaluation of reformed practice.

We believe that many of the observations regarding the teaching of mathematics at university level offered by the participants in our study would not necessarily sound new to researchers in mathematics

[10] The ones used in Nardi and Iannone (2003), a conference paper written in the earlier stages of the study.

education who are typically well versed in issues such as communication, interaction, catering for students' individual needs (Section 3.2.1), facilitating the transition from concrete to abstract mathematical ideas (Section 3.2.2), etc. The novelty and the value of the enterprise reported here, considering the comment above on the necessity to build bridges between the worlds of mathematics and mathematics education, resides in the fact that these observations originate totally from the practitioners of mathematics teaching themselves. As such, the aim of this paper has been two-fold: to introduce us to the pedagogical views of practising teachers of mathematics at university level and to offer an example of a type of the highly needed collaboration between mathematicians and researchers in mathematics education that is *possible*, *mutually appreciated* and *effective*.

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## Appendix A

In this appendix we wish to exemplify the data analysis process. We will show the transition from excerpts of students' writing to Episodes and to Stories. Consider for example the second cycle of data collection. The theme for this Dataset was "Mathematical objects I: the concept of limit across mathematical contexts". One of the questions included in the Dataset was (Fig. A.1), where we also include the lecturer's Notes on Solution, distributed to the students after marking was complete:

- The students' responses included in the Dataset are reported in Figs. A.2 and A.3.
- The participants were asked to consider the following issues prior to the group interview:

*Examples of issues to consider:*

1. The choice of $N$. As in the examples above many students chose $N = 1/\varepsilon^2$ instead of $1/\varepsilon^2 + 1$. What does this demonstrate with regard to their understanding of the inequalities and quantifiers in the definition? What evidence do we have about how students decide about $N$?
2. How do students interpret the request for 'writing out carefully the exact meaning of ...' in the question? Verbally? Symbolically? What do their choices imply with regard to their understanding of a formal definition?
3. In the above examples, the use of the definition of convergence is, for starters, adequate. Do your experiences with your students suggest otherwise?

The following interview transcript excerpt forms one of our Episodes. It refers to the above question and examples of student responses. It covers the period 7 m 31 s to 11 m 26 s of the first part (1 h 43 m) of the interview for this cycle.

**Question**

(a) Write out the exact meaning of the statement: "the sequence $(a_n)$ converges to A as n $\to\infty$",

(b) Prove using (a) that the sequence $b_n = 1 + 1/2 + 1/4 + \ldots + 1/2^n$ converges to 2.

**Notes on solutions**

5. "The sequence $(a_n)$ converges to A as n $\to\infty$", means:

$\forall\varepsilon > 0\ \exists N$ such that $n \geq N \Rightarrow |\, a_n - A| < \varepsilon$.

6. Given any $\varepsilon > 0$, chose $N = (1/\varepsilon^2) + 1$. Then:

$n \geq N \Rightarrow n > (1/\varepsilon^2) \Rightarrow 1/\sqrt{n} < \varepsilon$.

On the other hand

$|\, a_n - 2| = 1/\sqrt{n}$

so we have shown that

$n \geq N \Rightarrow |\, a_n - 2| < \varepsilon$

which proves that $a_n$ converges to 2 as n $\to\infty$.

Fig. A.1. One question from the second cycle's Dataset, accompanied by the lecturer's Notes on Solution.

B: I was a bit disappointed with this answer because the students are missing the point by . . . I see that .. carrying out some machine to . . . for the specific solution that has been applied to this problem. There are cases when you can do that . . .

E: They are asked to . . .. What else are they going to say? They have got to do something . . . They are . . .

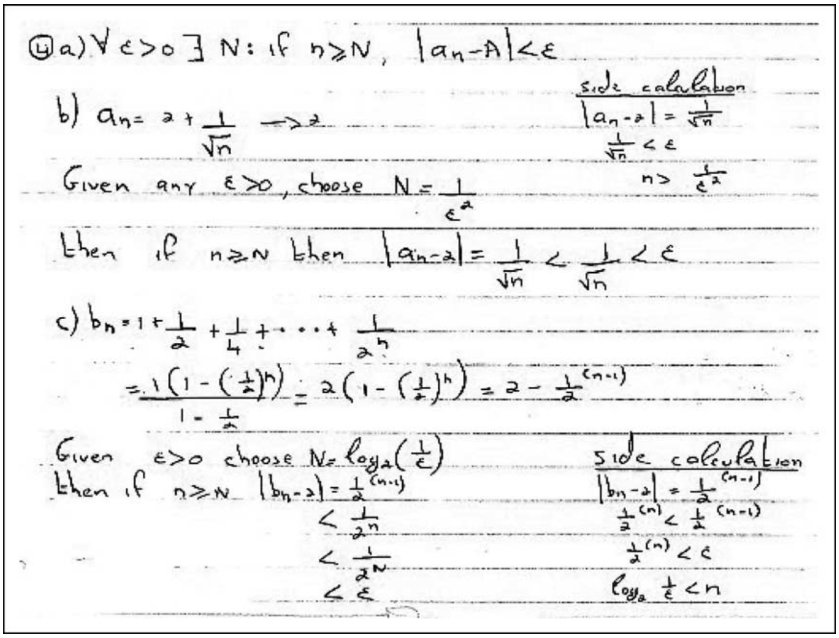
B: They could be approaching a problem where you couldn't do a solution in this way . . .

E: Ah, well . . . of course, but . . . if you are asked to prove that this series converges to 2 you have to know how to summit it.

D: Yes, there is a difference between having understood and . . .

E: Yes, you have to . . . which you can prove by induction. If they had to show convergence then of course they could have done many things.

B: Yes, but I was concerned that the student thought that $b_n$ was a number, and hence to be worked out, or a function and somehow this had to be shown explicitly to be converging to two in order to give the complete answer . . .



④a) $\forall \varepsilon > 0\ \exists N$: if $n \geq N$, $|a_n - A| < \varepsilon$

b) $a_n = 2 + \frac{1}{\sqrt{n}} \to 2$

side calculation
$|a_n - 2| = \frac{1}{\sqrt{n}}$
$\frac{1}{\sqrt{n}} < \varepsilon$
$n > \frac{1}{\varepsilon^2}$

Given any $\varepsilon > 0$, choose $N = \frac{1}{\varepsilon^2}$

then if $n \geq N$ then $|a_n - 2| = \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{n}} < \varepsilon$

c) $b_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n}$

$= \frac{1(1 - (\frac{1}{2})^n)}{1 - \frac{1}{2}} = 2(1 - (\frac{1}{2})^n) = 2 - \frac{1}{2}^{(n-1)}$

Given $\varepsilon > 0$ choose $N = \log_2(\frac{1}{\varepsilon})$

then if $n \geq N$ $|b_n - 2| = \frac{1}{2}^{(n-1)}$
$< \frac{1}{2^n}$
$< \frac{1}{2^N}$
$< \varepsilon$

side calculation
$|b_n - 2| = \frac{1}{2}^{(n-1)}$
$\frac{1}{2}^{(n)} < \frac{1}{2}^{(n-1)}$
$\frac{1}{2}^{(n)} < \varepsilon$
$\log_2 \frac{1}{\varepsilon} < n$

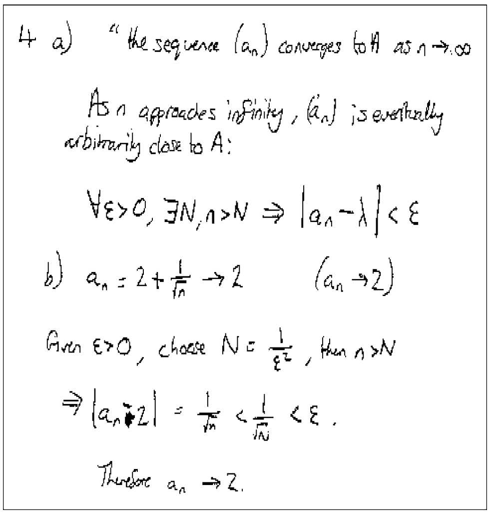
Fig. A.2. The response of student N (from the Dataset).

E: But I think . . . I think . . . How on earth can you prove . . . But how can you prove that the series is converging to something . . .

EN: But . . . at this time they don't know about Cauchy's sequences . . .

E: No . . . no . . . no.

EN: No . . .



4 a) "the sequence $(a_n)$ converges to A as $n \to \infty$

As n approaches infinity, $(a_n)$ is eventually arbitrarily close to A:

$\forall \varepsilon > 0, \exists N, n > N \Rightarrow |a_n - \lambda| < \varepsilon$

b) $a_n = 2 + \frac{1}{\sqrt{n}} \to 2$ $(a_n \to 2)$

Given $\varepsilon > 0$, choose $N = \frac{1}{\varepsilon^2}$, then $n > N$

$\Rightarrow |a_n - 2| = \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{N}} < \varepsilon$.

Therefore $a_n \to 2$.

Fig. A.3. The response of student MR (from the Dataset).

E: No!

(all laugh)

D: But even if they did. . .
E: But even if they did they somehow had to prove . . . somewhere you need to cheat . . . you need to know the formula for the geometric progression . . . or you need to prove it by induction. That $b_n - 2$ equals this, which you can get . . .

(0 h 9 m 9 s)[11]

D: And they did it on sheet one this year. Question 1 this year of sheet one was to show that that sum, by induction, is that.
B: Yes, but I was picking on the lines where if you had and answer many have found this sum and if you were given one that was converging, could you confirm that it is converging to that sum?
E: No, no, in general of course it would be completely helpless.
B: Ok, then I withdraw my comment.
E: Of course I think they have the formula wrong. . . but that is not so relevant!
B: I think it is . . .
E: Shouldn't it be $n+1$? Anyway . . .
B: $1-(1/2)$ all to the power $n+1$ . . .
E: No . . . In fact actually I mean, this is . . . this is something that I use deliberately. That I aspire to have odd moments in first year analysis where they meet with something they know. So I view this as . . . the first few lines of this . . . I view as something they know. In that they have seen it in the induction thing . . . and they have seen it at school. They will deny it but they still have seen it at school.
D: They actually have done this question?
E: They will confess under duress. And also it is . . . I don't know . . . it is. I am just . . . there is so little manipulative algebra in analysis which they are glad to do it when they can do it. . . . reassuring them . . . and there is so little of it . . . early on that I like to have some problems like this where they can feel they have done a calculation of the sort that they would recognise as a calculation and then it all goes to what they see as madness . . . it's nonsense . . . and . . . So, I mean . . . possibly there should be even more of these things . . . But it is artificial, I mean, Lecturer B is right. This is the one example where you can do this. And immediately afterwards you part company with this sort of approach, this way of thinking a series that you can compute what the sum is and . . . get closed almost to them . . .

This was then transformed into a narrative account (or *Story* as we have called it in Section 2). The two authors produced these narrative accounts independently. Below we present the narrative account written by the first author:

Narrative account from the Episode reported above, from the second Cycle of data collection.
Lecturer B says he is a little disappointed as he feels students are using some sort of machinery without understanding why, and this machinery will work only for this specific example. This strategy would not work for other more general examples. Lecturer E says that in this case they are asked to prove that it converges to 2, so they need to know how to sum the series, and if not they could prove it by induction. Lecturer B is still skeptical as he is concerned that the students thought that $b_n$ is a number "hence to

[11] The numbers in brackets are time codes: (h m s) = (hours minutes seconds).

be worked out". Lecturer E points out that otherwise you cannot solve the problem, and EN adds this is because they don't know about Cauchy sequences. Lecturer E observes that even if they knew about Cauchy sequences they would need to know the sum of this series, and Lecturer D comments that it was on the previous exercise sheet this year.

Lecturer B is again not convinced and points out that this strategy doesn't work in general. Lecturer E agrees with this: he observes that the student has got the formula wrong, but it doesn't matter. Lecturer B then concludes "Ok, then I withdraw my comment". Lecturer E now explains the rationale behind this exercise: "That I aspire to have odd moments in first year analysis where they met with something they know". And they have seen this sum in school, even if they will deny it. It is all done to reassure the students that they can actually do something this early on: "I like to have some problems like this where they can feel they have done a calculation of the sort that they would recognize as a calculation and then it all goes to what they see as madness". He agrees with Lecturer B that this is somehow artificial: "And immediately afterwards you part company with this sort of approach, this way of thinking a series that you can compute what the sum is".

Analytical triangulation through consultation of the two narrative accounts followed, as did agreement on the story's 'main focal points'. In this case these were:

1. Teaching practice: Lecturer E thinks that it is productive to show the students something they can do this early on, even if it introduces a strategy that is not going to work in the long run. It is in some sense a "confidence-building" exercise.
2. Not everybody shared this view and Lecturer B remains unconvinced.

Further scrutiny of the narrative accounts led to the classification of the 80 Stories in the 5 categories described in Section 2. The above story was placed in Category 2.

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