# PERSISTENT IMAGES AND TEACHER BELIEFS ABOUT VISUALISATION: THE TANGENT AT AN INFLECTION POINT

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*The role of visualisation in the mathematical reasoning teachers present to students may be influenced by several factors (e.g. mathematical, epistemological and pedagogical). Our study explores these potential influences through engaging teachers with tasks that invite them to: reflect on/solve a mathematical problem; examine flawed (fictional) student solutions; and, describe, in writing, feedback to students. Teachers are also interviewed. We discuss responses to one Task (which involved recognising a line as a tangent to a curve at an inflection point) of 91 teachers in order to explore the influence on the teachers' feedback to students of: (i) persistent images of the tangent line; (ii) beliefs about the sufficiency of a visual argument; and (iii) beliefs about the role of visual arguments in student learning.*

## INTRODUCTION

In the last twenty years or so the debate about the potential contribution of visual representations to mathematical proof has intensified (e.g. Mancosu et al, 2005), not least because developments in IT have expanded this potential so greatly. Central to this debate is 'whether, or to what extent, visual representation can be used, not only as evidence or inspiration for a mathematical statement, but also in its justification' (Hannah & Sidoli, 2007, p73). Recent works (e.g. Giaquinto, 2007) argue that visual means are much more than a mere aid to understanding and can be resources for discovery and justification, even proof. Whether visual representations need to be treated as adjuncts to proofs, as an integral part of proof or as proofs themselves remains a point of contention.

Within mathematics education the body of work on the important role of visualisation has also been increasing and has been focusing on issues as diverse as: curriculum development with an emphasis on visualisation (and often on related IT); mathematicians' perceptions/use of visualisation; students' seeming reluctance to engage (and difficulty) with visualisation; gender differences; links with embodied cognition; etc. – see Presmeg (2006) for a substantial review. Overall we still seem to be rather far from a consensus on the many roles visualisation can play in mathematical learning and teaching. So, while many works clearly recognise these roles, several (e.g. Aspinwall et al, 1997) also recommend caution with regard to 'the 'panacea' view that mental imagery only benefits the learning process' (p315). One of the aims of the study we report in this paper is to explore the role of visualisation with particular reference to the reasoning and feedback that teachers present to students. To do so we first introduce the study briefly and then discuss some of our data.

## THE STUDY AND THE TANGENT TASK

The data we draw on in this paper originate in an ongoing study in which we invite teachers to engage with mathematically/pedagogically specific situations which have the following characteristics: they are hypothetical but likely to occur in practice and grounded on learning and teaching issues that previous research and experience have highlighted as seminal. The structure of the tasks we ask teachers to engage with is as follows – see a more elaborate description of the theoretical origins of this type of task in (Biza et al, 2007): reflecting upon the learning objectives within a mathematical problem (and solving it); interpreting flawed (fictional) student solution(s); and, describing, in writing, feedback to the student(s).

In what follows we focus on one of the tasks (Fig. 1) we have used in the course of the study. The Task was one of the questions in a written examination taken by candidates for a Masters in Mathematics Education programme. Ninety-one candidates (of a total 105) were mathematics graduates with teaching experience ranging from a few to many years. Most had attended in-service training of about 80 hours. The first level of analysis of the scripts consisted of entering in a spreadsheet summary descriptions of the teachers' responses with regard to the following: perceptions of the *aims* of the mathematical exercise in the Task; mathematical *correctness*; *interpretation/evaluation* of the two student responses included in the Task; *feedback* to the two students. Adjacent to these columns there was a column for commenting on the means the teacher used (verbal, algebraic, graphical) to convey their commentary and feedback to the students across the script. The discussion we present in this paper is largely based on themes that emerged from the comments recorded in this column. In addition to the scripts we also collected data through interviewing a selection of the participating teachers: their individual interview schedules were based on the first level analysis briefly described above. Interviews lasted approximately 45-60 minutes.

The mathematical problem within the Task in Fig. 1 aims to investigate students' understanding of the tangent line at a point of a function graph and its relationship with the derivative of the function at this point, particularly with regard to two issues that previous research (Biza et al, 2006; Castela, 1995; Vinner, 1991; Tall, 1987) has identified as critical:

- students often believe that having one common point is a necessary and sufficient condition for tangency; and,
- students often see a tangent as a line that keeps the entire curve in the same semi-plane.

The studies mentioned above attribute these beliefs partly to students' earlier experience with tangents in the context of the circle, and some conic sections. For example, the tangent at a point of a circle has only one common point with the circle and keeps the entire circle in the same semi-plane.

Since the line in the problem is a tangent of the curve at the inflection point $A$ the problem provides an opportunity to investigate the two beliefs about tangency mentioned above – similarly to the way Tsamir et al (2006) explore teachers' images of derivative through asking them to evaluate the correctness of suggested solutions. Under the influence of the first belief Student A carries out the first step of a correct solution (finding the common point(s) between the line and the curve), accepts the line tangent to the curve and stops. The student thus misses the second, and crucial, step: calculating the derivative at the common point(s) and establishing whether the given line has slope equal to the value of the derivative at this/these point(s). Under the influence of both beliefs, and grounding their claim on the graphical representation of the situation, Student B rejects the line as tangent to the curve.

Year 12 students, specialising in mathematics, were given the following exercise:

'Examine whether the line with equation $y = 2$ is tangent to the graph of function $f$, where $f(x) = 3x^3 + 2$.'

Two students responded as follows:

**Student A**

'I will find the common points between the line and the graph solving the system:

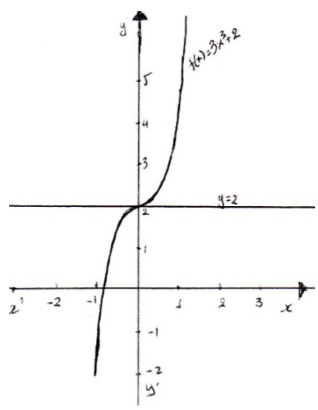
$$\begin{cases} y = 3x^3 + 2 \\ y = 2 \end{cases} \Leftrightarrow \begin{cases} 3x^3 + 2 = 2 \\ y = 2 \end{cases} \Leftrightarrow \begin{cases} 3x^3 = 0 \\ y = 2 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = 2 \end{cases}$$

The common point is $A(0, 2)$.

The line is tangent of the graph at point $A$ because they have only one common point (which is $A$).'

**Student B**

'The line is not tangent to the graph because, even though they have one common point, the line cuts across the graph, as we can see in the figure.'




a. In your view what is the aim of the above exercise?
b. How do you interpret the choices made by each of the students in their responses above?
c. What feedback would you give to each of the students above with regard to their response to the exercise?

Figure 1. The Task.

With regard to the Greek curricular context, in which the study is carried out, the Year 12 students (age 17/18) mentioned in the Task have encountered the tangent to the circle in Year 10 in Euclidean Geometry and the tangent lines of conics in Analytic Geometry in Year 11. In Year 12, they have been introduced to the tangent line to a function graph as a line with a slope equal to the derivative of the corresponding function at the point of tangency. Although in Years 11 and 12 the tangent is introduced as the limiting position of secant lines, this definition is rarely used in problems and applications.

One of the themes that emerged from the comments recorded in the spreadsheet with regard to the means the teachers used (verbal, algebraic, graphical) to convey their commentary and feedback to the students concerned the *beliefs (epistemological and pedagogical) of the teachers about the role of visualisation.* For example, with regard to the teachers' evaluation/interpretation of Student B's solution and feedback to Student B we explored questions such as: does the teacher turn the student away from the graphical approach (which may have led the student to an incorrect claim) and towards an algebraic solution in order to help the student change their mind about whether the line is a tangent or not? Does the teacher compare and contrast the algebraic solution to Student B's solution or do they proceed directly to the presentation of an algebraic solution? What types of examples/counterexamples, if any, do they employ in this process? What is the teacher's position towards Student B's grounding their claim on the graph? Etc.

In the course of this part of our analysis we noticed several influences on the teachers' responses: for example almost all teachers distinguished between (and often juxtaposed) Student A's *algebraic* approach and Student B's *graphical* approach. In almost all cases, in both scripts and interviews, the teachers included in their comments an evaluative statement regarding the sufficiency/acceptability of one or both approaches. And often they referred explicitly to their beliefs about, for example, the sufficiency/acceptability of the graphical approach; or about the role visual thinking may play in their students' learning. The teachers' responses also appeared significantly influenced by the mathematical context of the problem within the Task; namely, by their own perceptions of tangents and their own views as to whether the line in the Task must be accepted as a tangent or not.

At this point of our analysis we were somewhat surprised by the fact that 43 out of the 91 teachers supported Student B's claim that the line in the Task is not a tangent line – explicitly (25/91) or implicitly (18/91). In what follows we present examples from our analysis of the data from the 25 teachers who explicitly supported Student B's claim in order to examine the interplay between the teachers' *mathematical views* on whether the line is a tangent or not, *beliefs about the sufficiency/acceptability of the visual argument* used by Student B and *beliefs about the role of visual thinking in their students' mathematical learning*. We note that our examples originate in the scripts only, and not in the interviews, due to limitations of space.

## PERCEPTIONS OF TANGENTS AND BELIEFS ABOUT VISUALISATION

*Mathematical views on whether the line is a tangent or not*. Of the twenty-five teachers who explicitly accepted Student B's claim, ten rejected the line as a tangent without stating an argument (phrasing their responses as if this was obvious). The other fifteen stated that the line intersects with the curve without being its tangent either because point *A* is an intersection point but not a tangency point; or because it 'cuts across' the graph as student B argued. Three of these fifteen based the rejection on the fact that the line does not keep the entire curve in the same semi-plane. For example, *Teacher 101* claimed that 'it is not sufficient that the tangent line has only one common point, but it must keep the graph on the same side' and offered the graph in Figure 2. Taking a *local* perspective on Student B's '*cutting across*' argument the teacher also offers Figure 3 and says: 'the tangent could cut across the curve … the line is a tangent at $x_0$ [in Figure 3] although it cuts across the curve [at another point, *our addition*]'.

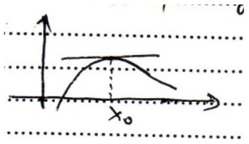



Figure 2.

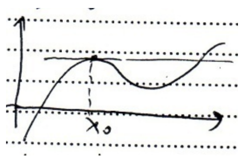



Figure 3.

*Beliefs about the sufficiency/acceptability of the visual argument used by Student B.* Of the twenty-five teachers, ten did not dispute this visual argument. Of these ten, eight made no reference at all to an algebraic argument. One teacher (*Teacher 81)* made some reference to both the 'algebraic and graphical methods' implying that she accepted the validity of both. She wrote that 'the aim of the exercise is that the students examine whether the line is tangent to the graph either graphically (if they can) or algebraically with the derivative' and later on observed that 'the exercise does not specify which way should be used to solve it'. (We return to this teacher's hint at the superiority of the graphical solution – 'if they can' – later in this section). Another one of these ten teachers (*Teacher 6*) set out with some reference to an algebraic argument that involved accepting the line; on the way, as she proceeded to a consideration of Student B's solution, she deleted the algebraic argument and concluded her response with agreeing with Student B.

The other fifteen teachers, while basing their inference on the graph and supporting Student B's claim, stated the need for supporting and verifying the claim algebraically (11 explicitly and 4 implicitly). These teachers, although they hinted at the algebraic solution for the justification of the answer, did not employ it in the argument they offered the students. As a result they did not confront the inconsistency of their statements. For example, *Teacher 97* wrote:

> To the first student I would say that it is not sufficient that the line and the curve have one common point but that the line must not split the graph as well. The derivative of the curve at *A*(0,2) must be equal to the derivative of *y*=2 at *A*(0,2). To the second student I

would say that his conclusion is correct and I would encourage him to give a better justified answer.

'Better justification' of a correct answer seems to sum up the views of these teachers. 'Better' is meant as:

- More general, feasible, useful:

> It is not acceptable to answer through graphical representations, because in a more complex case this approach is not feasible, *Teacher 1*

> […] even if a graphical understanding of functions is particularly useful, [Student B] should not forget that it is not always possible to use graphical representations and that he should learn to solve problems also algebraically, *Teacher 80*

- Offering a 'more rounded view of the problems', *Teacher 69*

- Accurate, because, for example, 'however helpful the graph may be, it is never totally accurate when done by hand', *Teacher 53*

- The graph not necessarily constituting a valid complete proof:

> If the exercise is asking for a proof it is better that the graph is accompanied also algebraically by the criterion for finding tangents through the first derivative and monotonicity, *Teacher 64*

In the above examples the teachers, while appreciating their students' employment of visualisation to reach a conclusion, are keen to stress that ultimately students are expected to demonstrate their capacity to complete the task algebraically. It is therefore possible that these teachers' embrace of the visual approach evident in Student B's solution is driven more by their belief in the gradual enculturation (Sierpinska, 1994) of the students into formal mathematical practice and their belief in the assistance that visualisation can provide towards reaching a conclusion (rather than a belief in the completeness of a graph-based argument). We cite below some evidence of these teachers' support for the employment of visualisation by their students.

*Beliefs about the role of visual thinking in students' learning.* A substantial number of the twenty-five teachers (nine) declared overtly their view of the graphical approach employed by Student B as evidence of 'conceptual' understanding. For example, *Teacher 68* applauded Student B and would say to him that he has a 'rounded way of thinking', 'his idea to investigate the problem graphically is very good', 'understands to a good degree what mathematical thinking is about' and 'carry on this way'.

Five teachers saw students' employment of the algebraic method as 'instinctively' driven by the conditioning students are subjected to in 'traditional' mathematics teaching. For example, *Teacher 4* wrote: 'Student A used the method mechanically' and 'Student B has a complete understanding of the problem'.

Eight teachers declared that the graphical solution is more 'quick and ready', 'clever' and 'not wasting any time', even 'natural' and 'real'. For example, *Teacher 48* claimed that 'through following a formal procedure we do not always reach correct results if we do not try at the same time to offer a 'natural' interpretation of the result' and would 'encourage Student A to always try to find the real dimension of the problem (e.g. through drawing the graph)'.

Several teachers, while expressing their appreciation for the graphical approach, stressed that students are not always at ease with it (see also *Teacher 81*'s comment earlier) and are often reluctant to use visualisation.

Overall many of the twenty-five teachers described the pedagogical role of graphical approaches as supporting students' use of their mathematical *intuition* and *imagination*. For example, *Teacher 69* stated that 'the aim of the exercise is to encourage students to combine their knowledge in mathematics with imagination in order to reach a result' and *Teacher 64* that 'Student B's answer is better, purely intuitive based, that is, only on the graph'.

It is perhaps interesting to see how some of these twenty-five teachers responded to Student A, with particular regard to that student's exclusive use of the algebraic method. Many attempted to balance their feedback to the students with regard to the approach they encouraged students to employ:

> It has been observed that many students have difficulty with algebraic manipulation while they are rather facilitated with visualisation, while for others the opposite applies. The teacher must encourage students to work in both ways, *Teacher 53*

In this spirit, and as we saw above, many teachers encourage Student B to work more algebraically. Analogously they encourage Student A to work also graphically:

> To Student A I would explain the mistake he has made and I would suggest one or two directions so that he tries to solve the problem again. I would also tell him to make the geometrical interpretation as this would help him. To Student B I would say that his answer is correct but that he would also need to justify it also algebraically. That is to make a synthesis of the algebraic and the geometric frame, *Teacher 85*

## CONCLUDING REMARK

The Task in Fig. 1 invited the teachers to offer feedback to two students one of which had used (incompletely) the algebraic method for deciding whether the line is a tangent and the other had used (incorrectly) a graphical representation of the problem. In this occasion the graph contained information that conjures up images that may lead to the rejection of the line as a tangent. About half of the teachers in our study appeared to get 'carried away' by this information – or, in Aspinwall et al's (1997) term, by these 'uncontrollable' images – and agreed with Student B's incorrect claim that the line is not a tangent. In the evidence we presented above the teachers appeared to get 'carried away' not simply by the images that they hold about tangents, conjured up by the graph in Student B's response, but also by a compelling tendency to support what they described as the more 'conceptual', 'imaginative' etc. approach of Student B. To them the mathematical problem in the Task offered an opportunity to convey their appreciation for the employment of visualisation.

However lack of awareness of the problems that certain imagery may cause, in this case the graphical representation of tangency at an inflection point, stands in the way of fulfilling the potential within the employment of visualisation.

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