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MATHEMATICS THINKING AND LEARNING AT POST-SECONDARY LEVEL

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INTRODUCTION

This chapter deals with research carried out on mathematics thinking and learning at post-secondary level. It tries to point out the evolution of research in this area since the first *Handbook* was published in 1992, its most important advances, their potential and limit for understanding and improving teaching and learning processes at this advanced level.

Synthesizing research advances in a particular area of mathematics education has always been difficult, due to the diversity of educational structures and cultures, and to the diversity of research paradigms. Regarding post-secondary education, the exercise of looking back 10 years could nevertheless seem easier than in other domains of mathematics education. Thanks to the existence of the International Group for the Psychology of Mathematics Education (PME), and its annual conferences, researchers from different origins reflecting on these issues established a pattern of regular exchanges and collaborative work. In 1991 this work led to the book edited by Tall: *Advanced Mathematical Thinking*, a good representation of the state of the art[1] at that time. Both the structure and the content of this book show that the predominant concerns then were cognitive ones: identifying cognitive processes underlying the learning of mathematics at advanced levels, investigating the relationships of these processes with respect to those at play at more elementary levels, and understanding students' difficulties with advanced mathematical concepts. Different theoretical constructs supported this research, such as the notions of *concept definition* and *concept image* (Tall & Vinner, 1981), the *process-object duality* (Dubinsky, 1991; Sfard, 1991), or the notion of *epistemological obstacle* due to Bachelard and developed in the didactic field by Brousseau (Brousseau, 1983). However, the book shows an evident common interest in visions of knowledge growth focusing on

[1] There were some significant exceptions: For instance, research carried out in this group did not consider the stochastic domain.

misunderstandings, cognitive conflicts, discontinuities, and hierarchies. Moreover, the research tended to concentrate on just a few mathematical domains: calculus and associated concepts, mathematical rationality, and proof. This vision, unsurprisingly, is reflected in the chapter that refers most directly to post-secondary level mathematics in the 1992 *Handbook*, "The Transition to Advanced Mathematical Thinking: Functions, Limits, Infinity, and Proof" (Tall, 1992).

The situation today is far from being the same, for a lot of reasons, linked both to the internal evolution of mathematics education as a scientific field and to external changes affecting post-secondary education. Research at post-secondary level was firstly influenced by the evolution of the dominant research paradigms from constructivist cognition and cognitive development towards sociocultural and anthropological ones (Lerman & Sierpinska, 1996). Sociocultural and anthropological paradigms were considered in the 1992 *Handbook*–see for instance chapter 20 by Schoenfeld who writes, "This cultural perspective is well grounded anthropologically but it is relatively new to the mathematics education literature" (p. 340). From that time, sociocultural and anthropological approaches have taken an increasing importance, and references to Vygotsky have tended to supplant those to Piaget. Research on mathematical learning at post-secondary level could neither escape this general influence nor ignore the central role these approaches give to the analysis of social and institutional practices in the understanding of knowledge growth. Research was also influenced by the emphasis these approaches put on semiotic mediations, and thus on the semiotic tools of mathematical activity. One can easily understand that such an evolution has a particular resonance in research on advanced topics: in advanced mathematics, students' relationship with symbolism becomes an essential feature of their relationship with mathematics. More recently, research has also been influenced by the increasing interest in the neuroscientific study of cognition and embodied cognition.

At the same time as these internal changes, the field was affected by a number of external changes, which have been analysed in the ICMI Study on the Teaching and Learning of Mathematics at University Level (Holton, 2001). For example, the following quotation comes from the ICMI discussion document that launched this Study and explained its rationale[2]:

> A number of changes have taken place in recent years which have profoundly affected the teaching of mathematics at university level. Five changes which are still having considerable influences are (i) the increase in the number of students who are now attending tertiary institutions; (ii) major pedagogical and curriculum changes that have taken place at pre-university level; (iii) the increasing differences between secondary and tertiary mathematics education regarding the purposes, goals, teaching approaches and methods; (iv) the rapid development of technology; and (v) demands on universities to be publicly accountable. Of course, all of these changes are general and have had their influence on other disciplines. However, because of its pivotal position in education generally, and its compulsory nature for many students, it could be argued that these changes have had a greater influence on mathematics than perhaps on any other discipline. . . . As a result of the changing world scene, ICMI feels that there is a need to examine both the current and future states of the teaching and learning of mathematics at university level. The primary aim of this ICMI Study is therefore to pave the way for improvements in the teaching and learning of mathematics at university level for all students. (ICMI, 1997).

All these evolutions have contributed to make the field of research on mathematics thinking and learning in post-secondary education much more diverse today than was the case 10 years ago. Consequently, trying to present a systematic survey of the results obtained in the last 10 years and of the different existing trends could result in a pointillist painting whose driving forces would remain invisible to the reader. Trying to avoid this trap while taking into account the state of the field obliged us to make some choices, and we acknowledge that the vision we present is a personal vision. Our main choice has been to reflect and question some major evolutions we perceive in the field rather than to give a comprehensive view of it. We also decided to approach these evolutions in two different ways: on the one hand by showing how classical research topics such as calculus or linear algebra have been partially renewed in the last 10 years and on the other hand by addressing two emerging research themes: mathematics in engineering courses, and stochastics.

Because this *Handbook* has no chapter devoted to technology, we would additionally like to present our own reflections on technology in this chapter. Consistent with the spirit of the *Handbook*, our choice is not to have a specific section dealing with technology, but rather to integrate the discussion of technology throughout the chapter.

[2] This discussion document was disseminated through different channels and can be found in the *ICMI Bulletin* number 43 (December 1997) which is accessible on the ICMI website (www.mathunion.org/Organization/ICMI).

The chapter is structured into four main parts. In the first part, we briefly recall the state of the art in the early nineties. In the second part, we consider evolutions that, in our opinion, exemplify how the internal evolution of the theoretical frames in mathematics education has influenced and is influencing research at post-secondary level. For this purpose, we mainly consider research in traditional domains such as calculus and linear algebra. In the third part, we consider evolutions more linked to external factors such as the evolution of the context of post-secondary mathematics education in its institutional, social, cultural, and technological dimensions and the evolution in the relative importance of the different mathematical domains. For this part, which did not have its exact counterpart in the previous *Handbook*, we focus on research carried out about the teaching and learning of mathematics in engineering courses, and research on probability and statistical learning. We hope that this choice helps us question theories and positions as regard mathematical learning and thinking that have generally been established having in mind more or less explicitly the mathematical education of "pure mathematicians" or mathematics teachers and their particular needs, and also that have focused on few and classical mathematical domains. This choice also obliges us to consider the role of technology in learning in a rather different way. Indeed, what is often at stake in these domains is not simply the use of technology for developing usual mathematical knowledge but the way in which technology is changing mathematical activity and understanding, including problem solving, proving, reasoning, modeling, and symbolizing. Mathematical and technological expertise and needs are there much more tightly intertwined.

Finally, in the last part of the chapter, we come back to more general issues, pointing out both the potential and limits of existing research for understanding and for helping improve the current situation of thinking and learning at post-secondary level, pointing out also some evident research needs that are apparent from the literature reviewed in the chapter.

THE EARLY NINETIES

As was mentioned in the introduction, the state of research on mathematical thinking and learning at post-secondary level in the early nineties is rather well represented in the book *Advanced Mathematical Thinking* published in 1991. A main theme of interest for researchers, at that time, was to characterize advanced mathematical thinking (AMT in the following) with respect to more elementary forms of mathematical thinking. It was also to clarify the mental processes that allow students to enter into AMT, and the difficulties students meet in developing such mental processes.

The Nature of AMT and AMT Processes

As pointed out by Dreyfus (1991) in his contribution to the book, characterizing AMT is not something as easy as "there is no sharp distinction between many of the processes of elementary and advanced mathematical thinking" (p. 26). As with more elementary levels, according to Dreyfus, the processes of AMT can be described in terms of representing, visualizing, generalizing, classifying, conjecturing, inducing, analyzing, synthesizing, abstracting, or formalizing. Referring to famous texts by Hadamard (1945) and Poincaré (1913), Dreyfus also pointed out the diversity of mathematical thinking modes, but differentiated essentially two main mathematical styles, according to the relative importance given to visualization and intuition, or to symbolic and analytic approaches.

Beyond these general considerations about AMT, the crucial point, as stressed by Tall in the final chapter of the book, was certainly the acknowledgement of the "thorny nature of the full path of mathematical thinking, so much more demanding and rewarding than the undoubted aesthetic beauty of the final edifice of formal definition, theorem and proof" (p. 251). For the authors, the gap between the logic of the mathematical edifice and the logic of cognitive processes explained the observed inefficiency of university teaching strategies based on the former, for the majority of students. This was evidenced at the time by the high rates of failure in fundamental courses, such as calculus, and also by the limited ability demonstrated later by those students who had passed the fundamental courses. The study of Selden, Mason, and Selden (1989) is from this point of view especially illustrative. These researchers presented students with problems[3] that could easily be solved with the techniques at their disposal but were not presented in the usual way. Not one student solved an entire problem correctly, and most of them could not do anything.

The gap mentioned above has been the motivation for several theoretical constructs, and we briefly present the most important ones in the following section.

[3] One of these problems was the following: Find at least one solution to the equation $4x^3 - x^4 = 0$ or explain why no such solution exists.

Concept Definition and Concept Image

These notions were introduced by Tall and Vinner (1981) and defined in the following way:

> We shall use the term of concept image to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built over the years through experiences of all kinds, changing as the individual meets new stimuli and matures. . . . As the concept image develops it need not be coherent at all times. The brain does not work that way. Sensory input excites certain neuronal pathways and inhibits others. In this way different stimuli can activate different parts of the concept image, developing them in a way which need not make a coherent whole. (p. 152)

One example of this lack of global coherence, often quoted in the literature, is the following: Students are asked first to compare 0.999. . . and 1, then to calculate the sum of the series: $\Sigma 9/10^n$. Many students answer that 0.999. . . < 1 to the first question whereas they correctly answer that the sum is 1 to the second. Several reasons have been given for explaining the first answer. They often rely on the process/object duality we will evoke later on: Students are bound to a process view of the symbolic notation 0.999. . . , and this view prevents them from seeing beyond the infinite process whose terms are all less than 1, the object "number 1" that results from it.[4] When asked the second question, they recognize a geometric series, and activating the formula for its sum they get the correct answer. Notice that this answer generally does not lead students to reconsider their answer to the first question, which is seen as a different one.

The concept image is generally at variance with the concept definition, and this was well evidenced at that time by research carried out on functions. Convergent results showed that many students, even when able to give a correct set-based definition of the notion of function, when asked to say if such or such object given by a discursive, tabular, symbolic, or graphic representation was or was not a function, gave answers at variance with their definition, and answers that could be different for the same object according to the semiotic representation used.[5]

Epistemological Obstacles

As mentioned in the introduction, other approaches had been developed in the eighties in order to approach the complexity of learning processes at post-secondary level and the distance between these and the current logical organization of mathematical knowledge. One of these approaches, also well represented in the AMT book, relied on the notion of epistemological obstacle, initially due to the philosopher Bachelard, and imported by Brousseau into the educational field. According to Bachelard (1938), scientific knowledge supposes the rejection of common knowledge. In the educational field, this notion has been introduced in order to better understand the status of students' errors and to acknowledge that some of these, generally the most resistant ones, result not from a lack of knowledge but from knowledge that has stabilized because of its efficiency. Let us stress that this knowledge can be social or cultural but that it can also result from school apprenticeship. Some researchers working at post-secondary level soon appropriated this notion in order to understand students' difficulties with advanced mathematical concepts such as the concepts of limit, derivative, and integral. Regarding the limit concept for instance, Cornu and Sierpinska first (Cornu, 1983, 1991; Sierpinska, 1985, 1987) and then Schneider (1991) evidenced the existence of epistemological obstacles related to

> The everyday meaning of the word limit, which induces resistant conceptions of the limit as a barrier or as the last term of a process, or tend to restrict convergence to monotonic convergence;
>
> The overgeneralization of properties of finite processes to infinite processes, following the continuity principle stated by Leibniz;
>
> The strength of a geometry of forms which prevents students from clearly identifying the objects involved in the limit process and their underlying topology. This makes it difficult for students to appreciate the subtle interaction between the numerical and geometrical settings in the limit process. (Artigue, 2001, p. 211)

Overgeneralization leads for instance to a belief that the limit of a sequence of strictly positive numbers is strictly positive, or that the limit of a sequence of continuous (resp. differentiable, integrable) functions is contin-

[4] Edwards (1997) proposed for instance an alternative interpretation based on the difference regularly observed in the treatment of the two equalities : 0.333. . . = 1/3 and 0.999. . . = 1. This would show that, when manipulating infinite decimal expansions, students do not refer to the definition they have been given but to their practice of division. This practice supports the first equality but cannot make sense of the second one.

[5] For instance, recognizing a constant function as a function if given by a graphical representation but refusing it this status if given by an algebraic expression without an explicit variable.

uous (resp. differentiable, integrable). The geometrical obstacle leads to a belief that whenever a sequence (F_n) of geometrical objects has, as a limit, an object A, all the magnitudes attached to the F_n have as a limit the corresponding magnitudes for A. As shown by Schneider, this geometrical obstacle often combines with an obstacle she labels the *heterogeneity of dimensions obstacle*, which underlies for instance the following reasoning concerning the computation of the area under a curve: When the size of the subdivision tends towards 0, each of the rectangles tends towards a segment. Thus at the limit, the total area under the curve which is the sum of the areas of the segments is necessarily 0, which is impossible![6]

Process/Object Duality

Another approach, which was to take increasing importance in the research community, was also already present: that based on the process-object duality. In the early nineties, this approach was associated with some particular names, especially those of Sfard and Dubinsky. In the AMT book, it was especially visible in the chapter written by Dubinsky where he introduced *APOS theory*. According to APOS, conceptualization begins with manipulating previously constructed mental or physical objects to form *actions*; actions are then interiorized to form *processes* that are then encapsulated to form *objects*. Finally, actions, processes and objects are organized more or less coherently in *schemas*, and "a subject's tendency to invoke a schema in order to understand, deal with, organize, or make sense out of a perceived problem situation is her or his knowledge of an individual concept in mathematics" (Dubinsky, 1991, p. 102).

Dubinsky's ambition was to isolate small portions in the complex structures of a subject's schemas and to give an explicit description of these, and especially to develop genetic decompositions of particular concepts such as induction and function, two examples he presented in the same chapter. What was also stressed by Dubinsky is the fact that a learner cannot successfully engage in AMT without developing an object view of the mathematical ideas at stake.

The 1992 *Handbook*

As mentioned earlier, Tall's chapter of the *Handbook* closely reflects the AMT book. Tall introduced the different constructs mentioned above together with illustrative examples, and we just refer here to the conclusion of the chapter where he gave his own characterization of AMT and proposed an agenda for research. As regards AMT, the crucial point, according to him, is the change in the relationship one develops with mathematical concepts: "To move to more advanced mathematical thinking involves a difficult transition, from a position where concepts have an intuitive basis founded on experience, to one where they are specified by formal definitions and their properties reconstructed through logical deductions" (p. 495), but he also stressed that "In taking students to the transition to advanced mathematical thinking, we should realize that the formalizing and systematizing is the final stage of mathematical thinking, not the total activity."

According to him, research has thus to investigate and understand the difficult cognitive changes and restructurings that this transition involves. Research evidences the existence of conflicts between students' intuitive views and formal mathematics; these have to be clarified through clinical interviews, and at the same time formal mathematics has to be itself placed into perspective as a human activity that attempts to organize the complexities of human thought into a logical system. This is the research agenda he proposed.

Beyond the work done under the AMT umbrella, post-secondary research is also present in the first *Handbook* through Schoenfeld's chapter "Learning to Think Mathematically: Problem Solving, Metacognition and Sense Making in Mathematics". Synthesizing different approaches to these issues coming from mathematics education and related fields, and the personal work he has carried out in universities on problem solving, reporting on differences between students' and experts' problem-solving behaviors, Schoenfeld introduced a framework for the analysis of mathematical cognition. This framework is organized around five dimensions: the knowledge base, problem-solving strategies, monitoring and control, beliefs and affects, practices. We would like to stress here the attention paid by Schoenfeld to the role played by the two last categories: beliefs and practices, and the links he establishes between these, as this attention shows the emergence of the more global and cultural views on cognition whose influence was to increase in the next decade.

Taking into account the specific theme of this chapter, we have focused in this brief summary on research on learning processes and mathematical thinking. Nevertheless we would like to mention that in the early nineties research on post-secondary mathemat-

[6] Note that recently Gonzalez-Martin and Camacho (2004) evidenced the existence in university students of a variant of this obstacle, when working on students' conceptions of generalized integrals. It is expressed in the conviction held by students that, for a positive function the integrals $\int_a^{+\infty} f(x)dx$ and $\int_a^{+\infty} f^2(x)dx$ have necessarily the same nature because to a finite area (resp. infinite area) there necessarily corresponds by rotation around an axis (here Ox) a finite volume (resp. infinite volume).

ics education was not restricted to this aspect. Teaching experiments were also developed consistent with these perspectives on learning. For instance, Tall used the notion of generic organizer[7] to build the software Graphic Calculus for introducing students to calculus concepts; Dubinsky and his colleagues developed the language ISETL and began to use it for teaching functions and algebraic structures; French researchers such as Artigue, Legrand, and Rogalski used the theory of didactic situations and the notion of scientific debate to develop (jointly with physicists) "didactic engineering" for the teaching of differentiation, integration, and differential equations (Artigue, 1991). Many of these developments incorporated technology as a generic organizer, as support for visualization and coordination between semiotic registers, or involved a programming language to support the interiorization and encapsulation of processes.

Thus, if we consider the field of research on mathematics learning at post-secondary level at the beginning of the nineties, there is no doubt that different theoretical constructs had been elaborated and worked out. These structured the didactic reflections of researchers about students' learning processes and difficulties, allowing researchers to better understand the failure of ordinary educational practices for the majority of students, to design alternatives to these and to test them. Nevertheless, the results obtained were limited to small areas of post-secondary education, both in terms of categories of students and mathematical subjects; the theoretical constructs were not generally integrated into more global didactic structures connecting both learning and teaching phenomena; they remained essentially cognitive constructs and, within this cognitive perspective, relied on the dominant constructivist epistemology. Both the evolution of the field and the evolution of the educational context have produced changes to this situation, and these we present and discuss in the following parts.

EVOLUTIONS DEALING WITH ALREADY-DEVELOPED RESEARCH AREAS AND PERSPECTIVES

In this part we present and discuss evolutions, focusing on those that can be easily connected to research areas and perspectives already mentioned above. First we consider the evolution of ideas about AMT itself. Second we consider the reinforcement and extension of existing approaches. We discuss the evolution of the process-object oriented approaches through the development of APOS theory, and the development of the proceptual approach. We also consider approaches that, starting from epistemological and/or historical analysis of mathematical knowledge, propose alternative categorizations for approaching the analyses of conceptualization. Third, we discuss the impact of some newer cognitive approaches, focusing mainly on the increasing influence of embodied cognition and of the linguistic approach developed by Lakoff and Nuñez. Fourth, we come to another dimension of the evolution that we see as a consequence of the increasing influence of anthropological and sociocultural approaches in the educational field. Cognition is there seen as something emerging from institutional practices, and understanding learning processes cannot be achieved without analyzing these institutional practices and identifying the norms and values underlying them. We show how these approaches complement the preceding ones for understanding the interaction between the individual and the collective in learning processes and supporting didactic design, and also for understanding issues that have become more and more crucial, such as the secondary/tertiary transition, or the relationships that new generations of students develop regarding mathematics and mathematical activity, and how these affect their learning of mathematics.

Evolution of Ideas About the Nature of Advanced Mathematical Thinking

The publication of the AMT book did not close the discussion about the nature of advanced mathematical thinking and its development. Critical reviews argued about what was referred to by the term *advanced* in the AMT book: mathematics? thinking? both? This was unclear, and the criteria used in the distinction between elementary and advanced mathematical thinking were not really convincing, according to the critics. From that time, the definition of AMT has been a question regularly addressed but unresolved. In the following, we have chosen to illustrate some of the associated discussions by referring to the work carried out since 1998 by a working group of PME-NA titled "The Role of Advanced Mathematical Thinking in Mathematics Education Reform" and the special issue of the journal *Mathematical Thinking and Learning* that

[7] Tall (1991) defined a *generic organizer* as "an environment that provides the user the facilities of manipulating examples (and, where possible, non-examples) of a concept. The word 'generic' means that the learner's attention is directed to certain aspects of the examples which embody the more abstract concept" (p. 187).

recently emerged from this work (Selden & Selden, 2005). As the editors explained in the introduction to this special issue, the working group "began by discussing such questions as what kinds of earlier experiences might help students make the transition to the kinds of AMT that post-secondary students are often asked to engage" and its interest "rather naturally metamorphosed into efforts at characterizing AMT and looking for seeds thereof that are, or could be, planted early in students' mathematical careers" (p. 3). The special issue evidenced the diversity of the answers offered within the group. For instance, starting from the fact that manipulating advanced concepts such as the concept of limit does not necessarily require advanced modes of thinking, and that students have difficulty finding referents for these abstract mathematical concepts in their familiar world, Edwards, Dubinsky, and McDonald defined AMT in the following way: AMT is "thinking that requires deductive and rigorous reasoning about mathematical notions that are not entirely accessible to us through our five senses" (2005, pp. 17–18). Harel and Sowder, for their part, defined AMT by referring to the notion of epistemological obstacle. Considering the three conditions imposed by Duroux and Brousseau (Brousseau, 1997) for epistemological obstacles: (a) to have traces in the history of mathematics; (b) to be a piece of knowledge producing valid answers in particular contexts, and invalid responses outside this context; (c) to "withstand both occasional contradictions and the establishment of a better piece of knowledge" (p. 34). Harel and Sowder claimed that "mathematical thinking is advanced, if its development involves at least one of the above three conditions" (2005, pp. 17–18).

These five authors proposed various examples in order to show the pertinence of the perspectives they developed but these examples, in spite of their interest, are insufficient to make the proposed definitions fully convincing. For instance, the first definition relies on the claim that some mathematical objects are directly accessible to our senses, an assertion that can be seriously discussed from an epistemological point of view. This definition also opposes rigorous and deductive reasoning to automatic reasoning or routine, which does not pay attention to the fact that most reasoning processes subtly intertwine routine and rigorous reflection. The separation between what is routine and what needs reflection changes for each learner according to the development of the learning process and even the academic institution where the learning takes place. Thus the operationality of this definition is questionable. As regard the second definition, the separation it artificially introduces between the three complementary conditions imposed by Duroux on the notion of epistemological obstacle, and the fact that these three conditions are given an equivalent role for qualifying AMT, seems difficult to sustain.

More than the attempts at defining AMT, which we do not consider as really successful, what we take from this special issue is the hypothesis made by most of the contributors, and especially Rasmussen, Zandieh, King, and Teppo, that AMT is not specific to a particular level of schooling. As these authors did, we would prefer to speak of *advanced mathematical practices.* Underestimating the relative nature of these, their dependence on institutional context, and linking them too much to formal mathematics can turn an idea that is a priori pertinent for analyzing learning processes into an obstacle. Our ambition in this chapter is to investigate how research carried out up to now at post-secondary level helps us to understand the ways students learn or do not learn, and could better learn mathematics, in the diversity of existing post-secondary institutions, without establishing a priori hierarchies of values among the different mathematics cultures these institutions offer. We need thus to be sensitive to the implicit values carried by the research constructs we use, and our review of AMT research leads us to be very critical in this regard about the construct of advanced mathematical thinking.

The Reinforcement and Development of Existing Approaches

Process–Object Duality

Expanding on the idea of process/object duality introduced above, we would like to focus on two trends: (a) the development of the APOS theory whose influence in research at post-secondary level has strongly increased in the last 10 years, thanks to the collaborative work of several groups of researchers and the existence of institutions such as RUMEC,[8] and (b) the development of the proceptual approach.[9]

The fundamental base of APOS remains the same as what was extensively presented by Dubinsky (1991). But, since that time, this approach has been used for building genetic decompositions of many different concepts taught at university level, for extending

[8] Research in Undergraduate Mathematics Education Community.

[9] We have chosen to focus on these two examples because they nicely illustrate what can be the long-term development of approaches based on the process-object duality, as far as these are widely used. A similar analysis could have been carried out starting from the theory of reification initiated by Sfard.

research based on APOS towards secondary mathematics education, and for elaborating and testing teaching designs based on the theory. APOS theory is a cognitive theory, and this obliges researchers to rely for teaching design on educational approaches complementing what APOS can offer. Thus the idea of cooperative learning has been linked to APOS, leading to a number of different projects (Dubinsky & Schwingendorf, 1997).

The fundamental basis of APOS has not changed, but one evolution is worth mentioning. It resulted from difficulties met by the researchers in satisfactorily explaining all the data they had collected. What was mainly at stake was the schema part of APOS. According to APOS (Dubinsky & McDonald, 2001, p. 277),

> a schema for a certain mathematical concept is an individual's collection of actions, processes, objects and other schemas which are linked by some general principles to form a framework in the individual's mind that may be brought to bear upon a problem situation involving that concept. This framework must be coherent in the sense that it gives, explicitly or implicitly, means of determining which phenomena are in the scope of the schema and which are not.

Data obtained in research concerning the chain rule, and then the properties linking the graph of a function and its derivatives, led to a reconsideration of this idea of schema, and to the incorporation of the triad introduced by Piaget and Garcia (1989) in order to better explain the construction of schemas. Incorporating the triad led to the introduction of three different stages in the construction: the Intra, Inter, and Trans stages. As explained in Dubinsky and McDonald (2001, p. 280),

> The Intra stage of schema development is characterized by a focus on individual actions, processes, and objects in isolation of other cognitive items of a similar nature.... The Inter stage is characterized by the construction of relationships and transformations among these cognitive entities.... Finally, at the Trans stage the individual constructs an implicit or explicit underlying structure through which the relationships developed in the Inter stage are understood and which gives the schema a coherence by which the individual can decide what is in the scope of the schema and what is not.

Taking as an example the function concept, one could say that at the Intra level, an individual tends to focus on functions seen as isolated objects and on the activities they perform on these; at the Inter level, he or she begins to make connections between functional objects, and to see how new functional objects can be created through these connections, to give sense to the idea of transformation of functional objects; at the Trans level, he or she can consider systems of transformations and the mathematical structures that emerge from these.

The second trend in process-object duality we evoke here is that developed by Tall. According to APOS, mathematical learning is achieved through the construction of mental actions, processes and objects, and their organization into schemas. Over the years, the distance between Tall's vision and APOS has progressively increased, and the model he proposes today is something quite different. According to him, cognitive growth of mathematical knowledge presents three main different paths corresponding to three different mathematical worlds (Tall, 2004):

> The first grows out of our perceptions of the world and consists of our thinking about things that we perceive and sense, not only in the physical world, but in our own mental world of meaning. By reflection and by the use of increasingly sophisticated language, we can focus on aspects of our sensory experience that enable us to envisage conceptions that no longer exist in the world outside, such as a "line" that is "perfectly straight." (p. 285)

This first world, which Tall names the *embodied world*, applies to the developmental path of Euclidean geometry from our perception of spaces and forms. The second path leads from enactive experiences with quantity and change to numbers, algebra, and calculus. Actions are there "encapsulated as concepts by using symbols that allow us to switch effortlessly from processes to do mathematics to concepts to think about" (p. 285), and Tall names the associated mathematical world the *procept world*. The third world is the *formal world*. Objects are there expressed in terms of formal definitions, and their properties deduced by formal proofs. They belong to mathematical structures defined through axiomatic systems. Geometry becomes axiomatic geometry; calculus becomes formal Analysis. Tall (2001) identified two different ways for the cognitive development of this formal world: natural thinking, which builds from concept imagery towards formalism on the one hand, and formal thinking, which builds from the concept definition, marginalizing imagery and focusing on logical deduction. According to him, individuals generally, according to the context, use one or the other way: "Natural thinking is appropriate for thought experiments that suggest possible theorems; formal thinking is appropriate for establishing formal proofs" (p. 235). But he also stressed that "formal thinking may also lead to 'structure theorems,' whose properties may be used to develop more subtle visual imagery, now enhanced by the network of formal relationships" (p. 235). Such distinctions are illustrated (Tall, 2001) using examples based on real numbers and infinitesimals. One can understand for instance the transitive property of the real order by relying on the image of the number line; and one can understand it as a consequence of the existence of a subset of positive elements P in a field satisfying the following axioms that make it an ordered field for the relation $x < y$ if $y - x$ belongs to P:

1) if x and y belong to P, $x + y$ and xy belong also to P,
2) Any x in the field satisfies one and only one of these properties: x belongs to P, $-x$ belongs to P, $x = 0$.

These two modes of understanding the same property formally expressed are obviously different.

In Tall's approach, relationships between processes and objects are seen in a much more dialectical way than in APOS, and symbolism contributes in an essential way to this dialectic through the notion of procept. This notion, first introduced by Gray and Tall (1994), expresses the fact that the same symbol can evoke both a process and the concept (i.e., object for Dubinsky) produced by this process; hence the name pro-cept, the condensation of process and concept. According to Tall (1996), "Function, derivative, integral and the fundamental limit notion are all examples of procepts. The theory of functions and Calculus can be summarized in outline as the study of the "doing" and 'undoing' of the processes involved" (p. 293).

Learning in this viewpoint means developing a flexible proceptual view. What this exactly covers is not something uniform, and Tall distinguishes between three categories of procepts according to their operational characteristics. Procepts in elementary arithmetic give direct access to the object they represent: The procept "4 + 8" represents both the process of adding the number 4 and the number 8, and the number 12 that results from this addition. Algebra is the domain of *template processes*: "$2x + 3$" can be seen as a process and as an object that I can for instance substitute for y in a procept such as "$2y^2 - 1$". As a process, it is not directly workable: it only tells how to get a numerical value each time a numerical value is given to x; hence the name template process. Procepts involved in calculus are generally of a third nature as they are associated to symbolizations that do not have the same algorithmic power, even if they can support some operational work, as for instance change of variables in definite integrals. Interpreting the symbol $\Sigma 1/n^2$ both as a symbol for a process and as a concept does not give means for practically operating with it, and for solving the mathematical problems that can be attached to it. In fact, developing a flexible ability for doing and undoing the processes involved in elementary calculus is certainly a rather complex and long-term construction not only dependent on the development of symbolic techniques and abilities. Tall insists in his most recent work on the role that can be played by enactive experiences and visualization, as mentioned above. The fact that enactive experiences can help students and even young students develop an intuitive sense of calculus concepts such as those of velocity and acceleration has been evidenced by different researchers for a long time (see for instance Kaput, 1992). As regards visualization, the software Graphic Calculus, developed by Tall in the 1980s (Tall, 1986), has provided a paradigmatic example ever since of what can be achieved for developing a first approach to calculus concepts without relying on formal definitions and proofs. It exploits the fact that fundamental notions in calculus such as those of continuity and derivative can be expressed in terms of *local constancy* and *local straightness*, and that the characteristics of computer visualization make it possible to "access" local properties through a finite number of zooms, and to "escape" in some sense the underlying limit process.[10] By working on these visual representations, one can thus explore the properties of mathematical objects that in fact result from an infinite limit process. Connections can be made with symbolic representations, and the interaction between these two different semiotic registers of representation can be used to support conceptualization and the development of procepts.

A major point of interest when thinking about post-secondary mathematics learning is obviously the transition towards formal thinking. Tall has often insisted on the radical change in perspective that this transition requires, and on its difficulty. According to him, only a few students truly enter this world but what is important to stress, and we will have the opportunity to come back to this point later on in the chapter, is that a lot can be achieved through mathematical work in the first two worlds mentioned above, the embodied and the proceptual.

We have summarized above some evolutions of approaches that, in the beginning, focused on the process/object distinction. The distinction between process and object levels of conceptualization is in-

[10] A comparison can be made here with research in stochastics in which simulations serve to create "microworlds" to explore abstracts concepts, such as sampling distributions or stochastic processes.

teresting for post-secondary learning because what students are asked to learn in university courses often requires an object level, whereas what they have learnt previously, even if they have developed some familiarity with the notions, does not ensure this object level, except for some limited set of instances in the best cases. In APOS, this initial distinction remained the core of the theory. It aims at representing some *vertical organization* of knowledge whereas the notion of schema takes care of the *horizontal organization* and of the underlying interconnections. But starting from the principle that any form of conceptualization obeys this pattern or, said in another way, that through the process/object duality we approach the epistemological core of knowledge growth in any mathematical domain is a very strong hypothesis that is open to discussion. We will have the opportunity to come back to this point in the last two parts of this chapter. In the research literature moreover, more evidence is emerging that establishing connections plays a fundamental role in knowledge development. In APOS, this dimension is accounted for by the schema (S) part of the model. But this part remains rather undeveloped compared with the APO parts. In our opinion, it is not by chance that researchers met problems with this schema part, which they tried to solve through incorporating the triad of Piaget and Garcia. A new hierarchy has been introduced, but the fundamental problem of the "extension of the concepts" cannot be solved just by introducing a new hierarchy. A fundamental question seems indeed to us the following: What does it mean to have built the concept of function as an object? What does it exactly mean to have an Inter, Intra, or Trans level of the function schema for students, and even for experienced mathematicians? Can we researchers give answers that are independent of the kind of functional objects we deal with, on the kind of operations we are asked to perform on these, on the kind of structures in which these objects are embedded?

The model proposed by Tall does not expose itself exactly to the same criticisms, but some points that are especially important when one thinks about post-secondary education seem to deserve much more research. The notion of procept recognizes the essential symbolic part of mathematical activity. Nevertheless saying that a symbol can flexibly refer to both a process and a concept is not enough for understanding the role played by symbols, and the mathematical work involving them, in conceptualization. This is especially the case in calculus and analysis because, as mentioned above, the operational value of symbols is not necessarily very high. Symbols are involved in techniques that can be seen as components of mathematical practices. The term *technique* is understood here with the anthropological meaning developed by Chevallard (1992): as a way of doing things, not necessarily something algorithmic. Techniques can be attributed both a pragmatic value linked to their effectiveness in producing results and an epistemic value linked to the way they contribute to the understanding of the mathematical objects they involve. The same can be said of the symbols that support techniques, and thus of procepts. Researchers need certainly to know more about the ways in which work with symbols can help develop the epistemic value of procepts. The second point we would like to make is about the transition towards formal thinking and the reconstructions this requires. More certainly needs to be known about this, and we think that complementary insights can be offered by other approaches, such as those we will describe in the next paragraph.

Epistemological Approaches

The perspectives described in this paragraph can be seen as complementary to the preceding ones. Even if the motivation of research is, as usual, the desire to understand students' difficulties and to develop didactic strategies helping them to overcome these difficulties, the base of the thinking is here more historical-epistemological than cognitive. This trend seems to us well illustrated by the book synthesizing educational research in linear algebra edited by Dorier (2000), and we will use this example in order to give an account of this position. We will come back then to what can be offered by such approaches to areas other than linear algebra, for instance calculus. As pointed out by Dorier in his epistemological work tracing the origin and evolution of the main concepts of linear algebra such as vector, vector space, linear independence, set of generators, basis, rank, dimension and duality, and linear transformations, the modern theory called *linear algebra* results from a long historical process covering several centuries, from the initial work on linear systems carried out by mathematicians such as Euler and Cramer around 1750 to the axiomatic theory of vector spaces, firstly formulated by Peano in 1880, which became universally accepted only 50 years later. In this sense,"the concept of vector space encapsulates, in a very elaborate product, the result of a long and complex process of generalization and unification" (p. 59).

The importance of this concept did not come just from the fact that it allowed the solution of new problems[11] but more from the fact that it opened new perspectives on old problems and productive connections between problems set up in different mathematical domains once these were recognized as instances of linear problems in adequate vectorial spaces. According to Dorier and his colleagues, the fundamental values of generalization and unification at the core of linear algebra are not easily understood, and the apparent simplicity of axiomatic structures is misleading. Understanding these values requires having already gained sufficient mathematical experience of linear problems in different contexts (geometry, systems of linear equations, sequences, differential equations . . .) and being able to adopt a reflective view in order to connect these different mathematical experiences. When introduced abruptly to the axiomatic structure of vectorial spaces, many students feel understandably overwhelmed with new definitions and vocabulary, immersed in a formalism they cannot really make sense of. Robert and her colleagues termed this the obstacle of formalism (Robert et al., 2000). Making sense of this formalism and understanding abstract linear algebra as more than a pure formal game requires one to establish reflective links with one's previous linear experience.[12] This contributes to make linear algebra, as pointed out by Dorier (2000),

> an "explosive compound" of languages, settings and systems of representation. There is the geometric language of lines and planes, the algebraic language of linear equations, n-tuples and matrices, the abstract language of vector spaces and linear transformations. There are the settings of geometry, of algebra, but also of graphical representations which allow a metaphoric use of geometry in higher dimensional spaces. There are the "graphical," the "tabular" and the "symbolic" registers of the languages of linear algebra. (p. 274)

As evidenced by research, teachers and texts constantly jump among these languages, settings, and semiotic systems as if conversions among these were obvious. To this complexity one must add the diversity of the associated reasoning modes. Sierpinska (2000) for instance distinguished three different reasoning modes that strongly intertwine in linear algebra: the synthetic and geometric in which mathematical objects are, in some way, directly given to the mind, which tries to grasp and describe them; the analytic-arithmetic mode in which objects are given indirectly by formulas or equations that make calculations with them possible; and the analytic-structural mode in which objects are also given indirectly, but this time through a set of properties. Again, Sierpinska showed that university teachers jump regularly without any precaution from one mode to another, leaving the responsibility for making connections to the student.

This is also an issue of the duality between the Cartesian and parametric points of view. Alvez Diaz (1998) has shown that, even if the conversion between parametric and Cartesian representations of subspaces can be, a priori, easily achieved thanks to algorithmic techniques in finite dimensions—those attached to the solving of linear systems—a flexible connection between these two points of view is hardly ever achieved by advanced students. Once more, looking back at history shows that the apparent easiness of such conversions is misleading. Understanding and mastering these took a very long time and was tightly linked to the long-term development of the concept of duality. Once more, an analysis of textbooks and teaching practices evidences the poor sensitivity of university educators to these difficulties.

The research on linear algebra offers views on learning processes that are complementary to the process/object research, emphasizing the specific epistemological status of concepts that have a particular generalizing and unifying role in the mathematical universe and suggesting that specific strategies have to be developed if one wants to have this epistemological status accessible to students. Of course, a more practical initiation to linear algebra can be organized, and this corresponds to an increasing tendency in post-secondary systems facing students' difficulties with abstract linear algebra. Powerful algorithmic techniques in linear algebra can give access to many conceptual ideas, as for instance shown by Uhlig (2002), and one can consider that this level of conceptualization is enough for many students. But research also shows that such an approach necessarily limits the modes of control that students can have on their mathematical work in linear algebra, and that when students are deprived of means of theoretical control on the

[11] When students are introduced to calculus concepts, for instance the concept of derivative, they quickly can understand what mathematical power they gain for solving a lot of problems linked to variation and optimization. Finding problems playing a similar role for showing the power gained through introducing linear algebra ideas is not so easy. As shown by the historical development of linear algebra (Dorier, 2000), axiomatic theory took a long time to be recognized as something essential, and this was achieved only when mathematicians used this theory for dealing with infinite nondenumerable spaces in functional analysis, something out of the range of university beginners.

[12] This is the rationale for different didactic strategies presented in the book.

techniques they use, they are easily trapped by "formal skids" as shown by Alvez Diaz (1998).[13]

The research also shows the essential status of the "connective dimension" in learning processes and highlights the diversity of the connections at stake: connections between settings and contexts, between points of view, between languages, between semiotic registers, and between reasoning modes. It thus confirms that cognitive flexibility is a crucial dimension of learning at these advanced levels and shows that this cognitive flexibility requires, to be adequately approached and understood, to go beyond the idea of conversion between different types of semiotic representation—the way in which it has often been analyzed in the research literature.

Coming back to the domain of calculus, one can identify similar phenomena in the transition towards formal analysis. When the fundamental concept of limit was formally defined in the 19th century, the main ambition was to establish the field of differential and integral calculus with solid foundations. This formal concept of limit can be given the status of "proof generated concept" according to the categories introduced by Lakatos (1976). Being sensitive to such mathematical concerns requires some particular mathematical culture, and developing this sensitivity in students certainly requires specific didactic strategies, as is the case for the notion of algebraic structure.

Whatever the way we express it, we touch here on the fact that different forms of conceptualizations can be a priori attached to the same concept, each of these being a coherent whole characterized by the experiences and problems that can be approached and made sense of, the techniques it offers for solving these problems, and a specific vision of mathematical rationality, and based on a specific level of formalization and symbolic manipulation. Conceptualization in any of these forms can a priori reach a high level of sophistication, as evidenced by history, and thus placing these too strictly in a hierarchy is not necessarily the most appropriate way of understanding mathematical thinking and learning, even if this is the general tendency. The idea of hierarchy leads us to see these different forms as corresponding to different cognitive levels and to think of cognitive growth as a transition process between these different levels. This is not necessarily the most appropriate metaphor for cognitive growth, as evidenced by research that shows that in expert practices different forms coexist, that university teachers for example jump so naturally from one form to another. If we do not adopt a strict hierarchical perspective, nevertheless other issues immediately emerge: issues of connection, relative importance, and respective role. These are different but not easier to solve, and certainly do not have uniform or universal answers. What researchers can nevertheless infer from both epistemological and educational research carried out up to now is that the connection between different forms of conceptualization is cognitively costly as it obliges to connect different ways of doing and thinking mathematically in a particular area, and to give each of these a particular role, and an importance that of course may vary according to the context. Research in geometry such as that carried out by Houdement and Kuzniak (1999), inspired by Gonseth's epistemological perspectives, can be considered as a step in this direction. Following Gonseth, they distinguish three main geometrical worlds: the worlds of practical geometry, natural geometry, and axiomatic structures, and they try to describe the characteristics that mathematical work presents in these different worlds. They show that each of these develops its own approach to geometry and rationality, and each has led to the development of sophisticated techniques and results, which are incommensurable between them. They also show that textbooks and elementary courses for preservice teachers jump without any precaution from one world to another (essentially between the first two), and that this fact makes it especially difficult for elementary teachers to understand the relationships between these different geometries, and the consequences for pupils' difficulties and curricular goals.

The Integration of New Cognitive Approaches

As evidenced by the research literature, for instance by the evolution of the AMT group of PME, educational thinking on learning processes is now influenced by different cognitive approaches. In this part, we consider the influence of neurosciences and theories of embodied cognition. Neurosciences have now informed the investigation of cognition in mathematics with research on perception and geometry (Berthoz, 1998; Longo, 2003) and on numbers and arithmetic, such as those by Dehaene (1996). Thanks to visual brain imaging, researchers can now localize those brain areas actuated in the solving of different

[13] For instance, Alvez Diaz shows that students customarily manipulate linear objects such as sets of vectors, matrices, linear transformations, linear systems, and determinants through numerical tables. These tables are, a priori, efficient semiotic instruments, but students tend to operate on the lines and rows of these tables automatically, without taking into account the mathematical meaning these operations have or not according to the object the table represents. This behavior generates "formal skids" leading to absurd results or contradictions that students generally do not notice.

mathematical tasks and generate a dynamic image of brain activity. The results obtained show some very interesting facts, for instance that the human brain seems to be genetically equipped with a sort of analogic counter that allows humans to globally compare quantities, and moreover that the brain areas involved in this are not the same as those used for counting and exact calculations. Research carried out with people who have brain damage shows also up to what point our numerical abilities are, at brain level, decomposed into a complex of localized processes dependent not only on the task but also on the precise context evoked in the task.

With some exceptions, these results have not concerned up to now advanced learning processes such as those considered in this chapter, but they do renew our attention to the dependence of learning processes on the biological condition of human beings. The increasing influence taken, in the educational world, by theories of embodied cognition reflects this attention. Embodied cognition, which began to develop in cognitive sciences in the eighties (Varela et al., 1991), investigates cognition as a physically embodied phenomenon realized via a process of codetermination between the organism and the medium in which it exists. Within this perspective, abstract understandings are grounded in bodily experience; an example frequently quoted is that of the concept of balance (Johnson, 1987) whose genuine meaning stems from our physical experience of bodily equilibrium and loss of equilibrium. Such bodily experience, according to the theory, gives way to perceptual-conceptual primitives called *image-schemata* that function as patterns and allow for the organization of experience. Their extension to different domains such as art (the "balancing" of colors in a painting), accountancy ("balancing" a budget), or mathematics ("balancing" an equation), which is generally a cultural phenomenon, leads to embodied concepts that are all grounded in this primitive physical experience. These extensions occur through conceptual mappings involving conceptual metaphors, hence the idea of mathematical knowledge as something whose source is metaphorical thinking (Lakoff & Nuñez, 2000).

The article published by Nuñez, Edwards, and Matos in the special issue of *ESM* "Teaching and Learning Mathematics in Context" (Boero, 1999) illustrates quite well how this perspective can be used to reflect on the learning of advanced mathematical concepts, such as the concept of continuity. Starting from the classical distinction between a natural idea of continuity characterizing a process without gaps (the vision of Euler of a continuous curve as a curve that can be drawn without lifting the pen) and the formalized idea of continuity due to Weierstrass, the authors showed that these two mathematical ideas are grounded in two very different conceptual metaphors. The first one grounds in what they call "the fictive motion metaphor," which they summarize by the following sentence: "A line is the motion of a traveller tracing that line" (p. 56). According to this cinematic vision, the curve is not a set of points but points can be put on it like milestones along a road, and points can move on it as travellers on a road. Students' and mathematicians' language is full of references to this cinematic vision of continuity, as also in everyday discourse. Metaphors can also be associated with the Cauchy-Weierstrass notion but these are radically different: "A line is a set of points"; "Continuity is gaplessness"; "Approaching a limit is preservation of closeness near a point." Points are in this case constituents of the line, and its continuous character that results from gaplessness is no longer evident. Being continuous for a function means that it preserves gaplessness: The image of a continuous set is a continuous set. The authors use this analysis for criticizing the usual teaching approach that establishes a clear hierarchy between these two conceptions of continuity and presents the $\varepsilon\ \delta$ definition as the one that captures the essence of the mathematical idea, which is in fact multiform. These authors reject a vision of learning about continuity that would see it as involving the rejection of the natural and cinematic conception and its replacement by the set-theoretical one, and the teaching strategies that more or less explicitly rely on such a vision. Of course, this does not contradict the fact that the set-theoretic conception, through the mathematical tools it provides, allows the solution of problems inaccessible to the cinematic conception. Even if expressed through another theoretical frame, it is interesting to point out here concerns very close to, for example, those expressed above concerning the different geometrical paradigms.

The influence of embodied cognition is also visible in the way technological issues are addressed today at post-secondary level.[14] In the early nineties, two main approaches in educational research featured at this educational level: the programming approach and the visual and multirepresentational approach. Educational research began to be sensitive to what could be offered by technological devices that simulate movement, thanks to the pioneering work by Kaput

[14] This is for instance visible in the way that Tall (2004) analyzes today the cognitive role of technological tools that he began to develop a long time ago, within other theoretical frames (as mentioned above).

and Nemirovsky with "Math Car" (Kaput, 1992), but this research concerned young students with limited mathematical knowledge. The impressive amount of research work carried out on representations (see the two special issues of the *Journal of Mathematical Behavior* edited by Goldin and Janvier, 1998, for a synthesis) and on the use of multiple representations for learning concepts such as the function concept, leading to what Confrey and Smith (1994) termed an *epistemology of multiple representations*, developed rather independently. As expressed by Borba and Scheffer in a special issue of *ESM* (2004), thanks to the development of technologies like CBR (Calculator-Based Ranger) the context today is changing and new perspectives are being offered to research. Body motion that was only peripheral to the multiple representation discussion becomes an essential component, and research on multiple representations takes another dimension through research concerning embodied cognition.

This connection, and the way it can influence research at post-secondary level is well illustrated in the *ESM* special issue in a paper by Rasmussen et al. (2004). These authors analyzed how three undergraduate students having already completed 3 semesters in calculus and taking a course on differential equations progressively made sense of an unfamiliar tool called the water wheel. Through this analysis, they tried to characterize how bodily activity and tool use can combine in mathematical learning, and how this combination can suggest alternative characteristics of knowing. The water wheel is a complex object. It consists of a clear, circular acrylic disc holding 32 plastic tubes around its perimeter. The disc is mounted on an axle and is free to rotate a full 360° and tilt between 0 and approximately 45°. In the center of the disc are two concentric clear plastic cylinders that contain a variable amount of oil that acts as damping for the system. Water from a bucket with a submersible pump (with adjustable flow rate) falls into several contiguous tubes on the "higher side" of the wheel. Each tube has a small hole at the bottom that allows water to drain out and be directed back into the bucket containing the submersible pump for a continual flow of water into and out of the tubes. When the wheel is tilted, gravity causes the wheel to rotate. When connected to a computer, an optical sensor collects data with real-time displays of angular velocity versus time, angular acceleration versus time, and angular acceleration versus angular velocity. Relying on the work developed by Nemirovsky, Tierney, and Wright (1998) on students' developing expertise with a motion detector, the authors' analysis takes into account the evolution of the students' relationship with the water wheel along several dimensions: becoming aware of how the tool may respond to certain aspects of one's kinesthesic engagement and not to others, acquiring an emerging awareness of what the tool does on its own, independent of one's action; gradually distinguishing which actions may affect the tool and the conditions that need to be in place so that it continues to remain sensitive to the actions; and developing a sense for what outcomes or results are possible, impossible, or difficult to obtain. Analysis along these dimensions helps the authors understand what kind of knowledge can be built through physically interacting with the tool, and how. This leads them to elaborate on a particular type of knowing called *knowing-with*, initially introduced by Broudy (1977). Knowing-with differs from commonly-identified forms of knowing such as knowing-that or knowing-how in that it involves at once the subjective feel and the objective sides of the experience and thus seems to the authors especially appropriate for describing the kind of cognitive construct that is at stake in the interaction with the tool. The language used by the students in this study shows indeed a very strong personal connection with the tool as if they were themselves becoming the tool, and this shapes the knowledge they develop about concepts such as angular velocity and acceleration in this environment.

Without any doubt, research like this attracts attention to learning processes that have been not sufficiently taken into account in post-secondary educational research up to now. The process raises interesting questions, shows intriguing phenomena, but is still in an exploratory phase. What kind of knowledge exactly is built in the sessions with the water wheel? How can students' utterances that are mainly of a qualitative nature, and often fuzzy, give way to something more analytical? How does knowing-with connect with the other forms of knowledge that develop at post-secondary level? How can it serve to understand and work with other mechanisms, other contexts? These seem today to be mainly open questions.

Investigating how qualitative knowledge that emerges from physical interaction with a calculator can be connected to more analytical forms of knowledge that are aimed at by educational institutions was a focus in the recent doctoral thesis by Maschietto (2002). This thesis addressed the transition between algebra and calculus, and the emergence of the dialectic interplay between, on the one hand, the point-wise or global perspective that characterizes the relationship that students develop with functions in algebra courses, and on the other hand, the local perspective that is attached to calculus and analysis. From a technical point of view, this emergence requires a reconstruction of the relationships with algebraic computations: The different terms of an algebraic expression are no longer given the same weight; their treatment depends on their respective orders of magnitude and relies on the fundamental idea of relative (or absolute, in nonstandard analysis) negligibility. The mathematical world of algebraic computations is thus deeply affected, at technical level and also in terms of strategies and control. Maschietto explored how these fundamental ideas can develop through adequate interaction with calculators, and the corresponding reconstructions may be promoted by appropriate didactic engineering, without entering the field of formal analysis. Maschietto relies on the potential offered by graphic calculators for visualizing local linearity (a potential that is well recognized today). The influence of embodied cognition is visible in the sensitivity she develops about the students' gestures and discourse as they zoom in and zoom out with the calculator, and about the personal and collective development of the metaphor of microlinearity. But what is added to these perspectives is a careful attention to the mathematical limits of visualization (it shows closeness, but not the order of the approximation that is essential here), a careful attention to the way the microlinearity metaphor can become an operational tool, and to the difficulties met by students in that operationalization. Students are very soon sensitive to the phenomenon of microlinearity that they recognize as an invariant. Moreover the distinction that they make between the two perceptive categories of straight and curved leads them to think that, beyond what they see, there is a more complex process involving infinity (something that is curved cannot become straight through a finite process). But mathematizing the situation in an operational way is not obvious and cannot be left to the students' responsibility. If not carefully dealt with, the microlinearity metaphor easily loses some of its essential attributes and only supports fuzzy discourse and knowledge; the necessary reconstruction of algebraic techniques takes time. Results show that fundamental and deep ideas and techniques of analysis can be developed in such a context rather early, but also that this requires a subtle design of the whole didactic engineering (Artigue, 1988), an adequate sharing of the mathematical responsibilities between the students and the teacher, and evident mathematical and didactic expertise from the teacher.

This research influenced by embodied cognition, by the attention it pays to cognitive processes and the situations in which they develop, and the possible "ecology" of these, leads us into the final section of this part devoted to more global research approaches, inspired by the increasing influence in the educational field of sociocultural and anthropological approaches.

The Integration of More Global Approaches

In this part, we consider approaches to mathematical thinking and learning at post-secondary level in which the consideration of sociocultural and institutional practices plays an essential role. According to the research culture in which it has developed or is developing, this consideration takes different forms and relies on different theoretical frames. These frames are often drawn from outside the field of mathematics education itself, as attested by the increasing number of references to Vygotsky, to Activity Theory, and to frames developed for understanding enculturation processes in various kinds of contexts. It relies also on frames developed inside the educational field. This is the case with the emergent approach developed by Cobb and Yackel (1996) in the USA that aims at connecting psychological constructivist and sociocultural perspectives for the analysis of classroom processes and the design of classroom experiments. This is also the case with the anthropological approach initiated by Chevallard from the early nineties (Chevallard, 1992; Chevallard & Bosch, 1999), which is clearly distinct from constructivist perspectives. The theoretical constructs developed by Godino and his colleagues in Spain (Godino, 2002; Godino & Batanero, 1998; Godino, Batanero & Roa, 2005) or by Cantoral, Farfan, and their colleagues in Mexico (Cantoral & Farfan, 2003) play for other researchers a similar role. These approaches all have their differences in detail, and the distance taken from constructivist and socioconstructivist perspectives may vary, but they share the common view that mathematical objects emerge from human practices, and that these practices are institutional and sociocultural practices. The term *institution* here has a very wide sense and includes any kind of formal or informal structure that organizes or conditions our social and cultural activities. The approaches also share the view that institutions develop with regard to any object that is recognized as a mathematical object a specific idea of what it means to know that object, thus defining a set of institutional norms for knowledge. So, the personal relationship one develops with mathematical objects emerges from the institutional practices and norms one has experienced in the different institutions where this object has been met. Within this perspective, understanding institutional practices and norms is sine qua non for the understanding of learning processes. Social and cultural mediations thus play a crucial role, and not by chance do the frames we are discussing pay so much attention to the semiotic dimension of mathematical activity, semiotic tools being an essential channel for these sociocultural mediations. In this part of the chapter, we would like to show the complementary insights these approaches offer to the understanding of learning processes and students' difficulties. Different examples will help us to illustrate what is offered and to show the diversity of the research developed within these perspectives.

In all the perspectives considered here, individual knowledge emerges from sociocultural practices. Both individual and collective knowledge develop in the classroom, in a subtle intertwining, and one cannot understand individual cognitive development without considering its collective counterpart. Our first example illustrates this dimension, showing one perspective developed within the emergent approach for analyzing the collective construction of knowledge.[15] The research (Stephan & Rasmussen, 2002; Yackel, Rasmussen, & King, 2000) deals with the teaching of differential equations and was carried out in the frame of a 15-week reformed course that tried to benefit from technology for emphasizing graphical and numerical approaches to the topic. From a theoretical point of view, it relies, as mentioned above, on the emergent approach, which considers that learning is both an individual and social accomplishment with neither taking primacy over the other. The design of the instructional sequence was inspired by the theory of realistic mathematics education, initiated by Freudenthal (1991) and further developed by the Freudenthal Institute (Gravemeijer, 1999). According to this, learning trajectories are built, giving students the opportunity to create meaningful mathematical ideas as they engage in challenging mathematical tasks. The development of knowledge is organized around cycles of horizontal and vertical processes of mathematization (Treffers, 1987). Horizontal mathematization refers to "formulating a problem situation in such a way that it is amenable to further mathematical analysis" whereas vertical mathematization "consists of those activities that are grounded in and built on horizontal activity" such as "reasoning about abstract structures, generalizing and formalizing" (Rasmussen et al., 2005, p. 54). Further, in this research the collective development of knowledge is approached through the analysis of argumentation, by using Toulmin's (1969) model of argumentation. According to this model, the core of an argument consists of three parts: the data, claim, and warrant. Briefly speaking, the data provide evidence for the claim, and the warrant explains how the data leads to the claim. Moreover, in case the validity of the warrant is challenged, the presenter must provide a backing to justify why the warrant, and therefore the core of the argument, is valid. Progression of collective knowledge is associated to the development of taken-as-shared knowledge, and this is achieved through a detailed analysis of the evolution of collective argumentation during the course. The researchers consider that mathematical ideas become taken-as-shared in the classroom community when

> either (1) the backing and/or warrants for an argumentation no longer appear in students' explanations and therefore the mathematical idea expressed in the core of the arguments stands as self-evident, or (2) any of the four parts of an argument (data, warrant, claim, backing) shift position (i.e., function) within subsequent arguments and are unchallenged. (Stephan & Rasmussen, 2002, p. 462)

This allows the researchers to develop an interesting analysis of the progression of the classroom knowledge, organized around the emergence and development of six mathematical practices: predicting individual solution functions, refining and comparing individual predictions, creating and structuring a slope field as it relates to prediction, reasoning about the function P (in $P'(t) = f(P)$) as both a variable and a function, creating and organizing collections of solution functions, and reasoning with spaces of solution functions. The researchers stress that the analysis shows a nonsequential development over time of these different practices. For instance, the idea that, for an autonomous differential equation,[16] the phase portrait is invariant by horizontal translation began to emerge in the second session (not of course articulated in the expert language we use here) and went on being discussed until the 11th session. These results are of course linked to the choices made in the design of the instructional sequence regarding the learning trajectory, the precise tasks proposed to students along this trajectory, and the sharing of mathematical responsibilities between teachers and students. The mathematical practices developed here contrast with the practices usually adopted in differential equation courses and offer evidence that teaching designs that are more respectful of the epistemology of the field, both meaningful for students and reasonably ambitious[17] from a mathematical point of view, can exist. The emergent approach, by the attention it pays to the

[15] Another interesting dimension of this research we just mention here deals with the progressive development of sociomathematical norms in this reformed course.

[16] A differential equation $y' = f(y,t)$ is said autonomous if the function f does not depend on t. This reform course only dealt with autonomous differential equations.

[17] The course for instance approaches the dynamics of differential equations depending on parameters and introduces and discusses bifurcation diagrams.

connection between the individual and the collective, helps to understand the essential role played by the classroom community in the individual development of knowledge. By the attention it pays to social norms and sociomathematical norms, it also makes clear that new practices cannot be developed without removing some of the constraints induced by usual norms, and thus raises the issue of the institutional viability of innovative practices.

The second example we present deals with a crucial issue: that of the transition between secondary and tertiary education. This transition is widely known as a serious problem, and the difference between the mathematical cultures of secondary school and college/university is generally admitted as an essential source of it. These two cultures are moreover nowadays generally regarded as two cultures moving on roads increasingly distant, which can only make the transition harder and harder. We are thus interested in looking at this problem from an anthropological or sociocultural perspective. This was the perspective adopted by Praslon (2000) in France, in his thesis devoted to the notion of derivative at the transition between high school and university. Praslon used the anthropological approach of Chevallard and the associated conceptual tools in order to explore the characteristics of the two cultures (limited nevertheless to the case of science-focused students both at high school and university). In this approach, as stated above, knowledge emerges from practices. Practices are analyzed through the notion of *praxeology*, a complex that includes types of tasks, ways for solving tasks called techniques (not necessarily algorithmic techniques in the usual sense), a discourse that explains and justifies techniques called technology, and theories that organize and structure the technological discourse. This led Praslon to characterize the praxeologies involving the notion of derivative in the two different institutions that he considered. This characterization led to interesting results. He showed that, contrary to what is often said, the secondary-tertiary transition in this area is not a transition between informal and formal mathematics, between intuitive and rigorous approaches. Rather the transition is more an accumulation of "micro-breaches," which affect the balance between the tool and object dimensions of the derivative, the balance between the study of particular objects and objects defined by general conditions, and the balance between algorithmic techniques and techniques that have more the status of general methods to be adapted to each particular case. These shifts also result from the increased autonomy given to students: as regards the choice of appropriate settings, appropriate semiotic registers, and the connections and changes made between these during the solving process, and more globally as regards the overall development of the solving process (similar tasks often appear in the two institutions, but the number of intermediate questions in a problem is far from being the same). They also result from the incredible diversification of tasks that occurs. From a culture organized around the mastery of a restricted number of tasks that can become reasonably familiar, and in which the progression in complexity is carefully managed, students pass into a culture in which diversity is the norm and the greater number of new notions to be covered in the same time period makes routinization of practices much more difficult. Once more it is evident that cognitive flexibility is an essential criterion for success. In order to make both students and university teachers sensitive to the differences between the two cultures, Praslon has created a set of tasks that, according to his results, are situated today in the "gap" between the two cultures and reveal the main facets of the differences between these. Purposely, none of the tasks requires solution by formal analysis. They are designed to be proposed to the students before their entrance to university or at the beginning of the academic year, and the students' work discussed with university teachers.

The anthropological framework is also involved in the *instrumental* approach initiated by French researchers for addressing issues linked to the integration of Computer Algebra Systems (CAS) in mathematics education (Guin, Ruthven, & Trouche, 2004). Combining Chevallard's anthropology with perspectives coming from cognitive ergonomy (Rabardel, 1995), the instrumental approach analyzes how mathematical praxeologies are affected by the use of CAS in their technical and technological components. This approach leads to specific attention on what is called *instrumental genesis*, which is the process in which there is a transformation of a tool (here a CAS) into a mathematical instrument, either for an individual or for an institution. The results obtained (Guin et al., 2004) show the complexity of this process, which for a long time has been underestimated in educational research dealing with technology, and the deep extent to which learning processes intertwine mathematical knowledge and knowledge about the tool itself. The sensitivity to institutional aspects allowed by the anthropological frame leads also to a better understanding of how a teacher's professional work is affected by the integration of such tools and offers a vision of these changes quite different from what is generally offered by the literature. We will not enter into more details here regarding this approach, whose constructs have been elaborated and used in the context of secondary education up to now (see Guin et al., 2004, for such details). However, the results obtained on mathematical topics such as calculus that are generally taught primarily at post-secondary level lead us to think that it can be of interest in the future for research involving technology, and especially professional technology[18] as is the case for CAS, at post-secondary level. Its constructs could certainly also be usefully connected, to those developed by Nemirovsky and his colleagues mentioned above, as the two approaches share the same fundamental idea: technology *cannot* be considered only as a kind of educational assistant. It deeply shapes what we learn and the way we learn it, and an efficient integration of technology in mathematics education has to take this seriously into consideration, in terms of institutional practices, values, and norms.

In the preceding, we have approached the idea of culture in terms of practices and praxeologies. Other categorizations can be used. Those introduced by Hall (1981) have been used for instance by Sierpinska (1989, 1994) and more recently Nardi (1996). Hall recognizes three types of consciousness, three types of emotional relations to things: the formal, the informal, and the technical. In the context of mathematical culture, the formal level corresponds to beliefs around what is mathematics about, what are the legitimate tools and methods for mathematical work, and so on; the informal level corresponds to schemes of action and thought, unspoken ways of doing things or thinking that result from experience and practice, what is also often called "tacit knowledge"; and the technical level corresponds to the explicit part of knowledge, to the techniques and theories. Approaching learning as an enculturation process leads researchers to try to understand how these three levels and the interactions between these can develop in a given culture, and why. Sierpinska for instance first used this approach in reflecting on the long-term program of research she had developed about the learning of the concept of limit. In this program, the determination of epistemological obstacles, further complemented by the dual notion of "act of understanding," had played an essential role. Thinking in terms of Hall's categories, Sierpinska inclined to ask herself what the cultural situation of epistemological obstacles can be. According to her, these can be situated both at formal and informal levels, but not at the technical level.

Nardi (1996) used Hall's categories in order to understand the tensions between the mathematical culture of students entering the university and that of the university. She also looked at the ways enculturation progresses, taking into account not only the technical level as is usually done but also the formal and especially the informal level, through the accumulation of experience shared with the expert (here a tutor) and in the process of appropriation by the internalising imitation of the expert's cultural practices. The data she collected led her to focus on concept image construction and on formalization in order to analyze the enculturation/cognitive processes at stake. She pointed out the tension existing between the informal-intuitive-and-verbal mode of thinking on the one hand, and the formal-abstract-and-symbolic mode on the other, and the difficulties students meet with the mechanics of formal reasoning. These tensions are especially visible in calculus, but Nardi showed that the exercises proposed to students, instead of helping them overcome the epistemological obstacles they face, can reinforce them: for instance when exercises constantly expose students to infinite sums that can be broken up and rearranged as finite sums, reinforcing their belief that infinite sums can simply be treated in the same ways as finite sums.

The two examples just presented focus on the understanding of students' learning of mathematical concepts. But students' learning is also affected more generally by the cultural practices they are part of, and by the visions of mathematical activity and mathematical learning these practices induce. As we noted above, such concerns were already present in the first *Handbook*, in the chapter written by Schoenfeld. In our opinion, even if research has remained sensitive to these issues, knowledge in this area did not substantially improve during the last decade. Today, nevertheless, one can think that changes in the context of university teaching—massification of tertiary education and the resulting diversity of the student population, increasing student disaffection for mathematics and science—can stimulate research. Moreover, the development of cultural approaches over the last decade better equips researchers for approaching the difficult questions at stake. From this point of view, the article by Zevenbergen (2001) in the ICMI Study on teaching and learning mathematics at university level is an interesting example. Zevenbergen's particular concern is about equity and the characteristics of mathematical practices at university that can be seen as excluding some groups of students, preventing them from learning and succeeding. His interest for this issue seems linked to the expansion of the higher education sector and to the resulting increased diversity of students. His approach is cultural and especially relies on sociolinguistics (Halliday, 1988). Language is seen as a form of cultural capital, and the role of language in

[18] That is, a CAS is software created to satisfy the needs of professional users of mathematics, rather than software created for the needs of mathematics learners.

the access to mathematics is seen as something essential. Hence the interest in the analysis of the linguistic practices in mathematics and in the understanding of their potentially excluding role for new categories of students entering the university: those from the working class and those who are not native English speakers. As pointed out by this author, research within these perspectives has been mainly concentrated up to now on elementary and secondary school. Very little has been done at post-secondary level.

Another piece of research that illustrates how the evolution of theoretical approaches can serve to renew research approaches is the recent article by Castela (2004). This research analyzes how tertiary institutions influence students' personal study, and thus learning, by comparing two different types of tertiary institutions that exist in France: specific courses (known as CPGE) that prepare students for entry to the most prestigious business and engineering schools, and university courses. The stimulus for her research was the observed discrepancy between the respective rates of success of students coming from these two types of institutions at the national competition for secondary teachers (CAPES), after 1 year of common preparation. From this, she investigated the characteristics of the mathematical cultures in the two institutions and how these influenced the students' vision of mathematics learning and personal achievement in mathematics, and how this achievement is more or less sufficient for the purposes of entering the CAPES competition. She showed that the type of task proposed at the CAPES (long problems covering extensive mathematical domains) requires specific competences in terms of strategy and structuration of problem solving that are not really taught. Then she tried to match these requirements with the ways that the two categories of students usually work. She for instance showed that CPGE students have a rather operational vision of mathematics and are sensitive to the need for competence in strategy and structuration. University students however perceive mathematics more in terms of content and put more emphasis in their work on lectures. She identified three different approaches to problem-solving in students' practices: a drill-and-practice approach, a reproductive approach, and a transferential approach. The first approach is more representative amongst university students, the last more amongst CPGE students. By deploying the tools provided by didactic anthropology, she then showed how these different attitudes can emerge from the adaptation of students to the institutions in which they study.

We find it interesting to connect these results with those obtained by Lithner, in his observation of students' functioning with textbook exercises (Lithner, 2003). Lithner used three categories in order to qualify the forms of reasoning used by students in task solving:

plausible reasoning (PR) if the argumentation developed

i) "is founded on intrinsic mathematical properties of the components involved in the reasoning, and
ii) is meant to guide towards what probably is the truth, without necessarily having to be complete or correct" (p. 32),

established experience (EE) if argumentation

i) "is founded on notions and procedures established on the basis of the individual's previous experiences from the learning environment,"
ii) [the same as for plausible reasoning] (p. 34),

identification of similarities (IS), if

i) "the strategy choice is founded on identifying similar surface properties in an example, theorem, rule, or some other situation described earlier in the text,
ii) the strategy implementation is carried out through mimicking the procedure from the identified situation." (p. 35)

Even if the academic results of the three categories of students observed by Lithner are quite different, clear similarities appear in their functioning. Almost all of the students' homework time seems to be spent on exercises. IS reasoning dominates strongly, and the evaluation is done by comparing with the textbook solutions section. He also showed that, due to the organization of the textbooks used and to the exercises they propose, most exercises can be solved by IS reasoning. In Lithner's experiment, more elaborated forms of reasoning became necessary when students made errors in the implementation of their IS strategies, but, even in this case, PR remained very limited. What is observed suggests that the forms of work based on memorizing and mimicking can be considered by the students as efficient for passing exams, and he sees these as "unintended by-products of their mathematical instruction" (p. 54).

In this part of the chapter, we have focused on what we perceived as major evolutions of different theoretical approaches linked to both internal and external factors in the field of mathematics education. The choices we made for our presentation helped us structure it, but perhaps we have too strongly opposed the sociocultural and anthropological approaches to the preceding ones. The reality of research is much more complex. Complementarity is certainly a more appropriate term than opposition. Many researchers combine several frames in the same research project.

For instance, Nardi (1996) combined a cultural approach and the use of process/object duality, whilst Praslon (2000) combined an anthropological general approach with different cognitive perspectives used in educational research about calculus. Favoring certain approaches nevertheless shapes the problematics and methodology of the research, and through these the kind of results that one can access and the way they will be expressed. Hence there is an explosion of notions and terms that is not easy to make sense of, but links and partial translations are often possible, as we have also tried to show. In the next part of the chapter, we broaden our discussion to new ideas and new coherences, serving to challenge and complement those already presented, which will provide further evidence of the complexity of the issues at stake, and the fact that any single perspective can only approach this complexity very partially.

EVOLUTION THROUGH THE DEVELOPMENT OF NEW RESEARCH AREAS

In this part, we turn our attention to two areas of research that lie outside the core mathematical domain generally considered by advanced mathematics education: research on the teaching and learning of mathematics in engineering courses, and research on learning probability and statistics. In these two areas we hope to illustrate the effects of the five influences alluded to by the ICMI Study (Holton, 2001):

> (1) the increase in the number of students who are now attending tertiary institutions; (2) major pedagogical and curriculum changes that have taken place at pre-university level; (3) the increasing differences between secondary and tertiary mathematics education regarding the purposes, goals, teaching approaches and methods; (4) the rapid development of technology; and (5) demands on universities to be publicly accountable.

Although public accountability is a generally political issue that may seem to lie beyond the interest of researchers, nevertheless it impacts quite directly in several ways, largely due to influence (1), the growing proportion of young people in tertiary education. Public scrutiny of "teaching quality" in universities has become rather common in the last 15 years, as increasing amounts of public money are spent on greater numbers of students. But also there is the case that as student numbers grow then the funding mechanisms have changed in many countries, with students themselves paying growing proportions of their tuition fees. Thus there is another accountability to be met as students come to expect a good "service" for their money, and perhaps hold stronger opinions than before about the education they expect, setting up a tension with the education that their teachers believe is appropriate for them.

Statistics is one of the most widely taught topics at university level, where many service course students meet advanced stochastic thinking without any prior or concurrent experience of advanced algebra or calculus. At the same time, statistics is separating from mathematics as an academic discipline (e.g., in the training of mathematicians and statisticians; research journals and associations) and is taught by teachers with a variety of backgrounds, rarely by pure mathematicians. As regards secondary education, stochastics is receiving increasing attention in recent curricula (e.g., National Council of Teachers of Mathematics [NCTM], 2000), where it is considered as an essential component of mathematics education, both due to its instrumental role to understand other disciplines and as a vehicle to develop critical reasoning and democratic values.

In the development of mathematics courses for engineering, the main trend comes from the technological influence, particularly stimulated by changes within professional engineering practice. The traditional teaching approach, based on having students spend much time developing fluency with pen-and-paper mathematical techniques, is giving way to students learning how to use computer software (such as spreadsheets or computer algebra systems) to carry out mathematical calculations. On the one hand, this is liberating and empowering, because students can more quickly become engaged in solving realistic engineering problems. On the other hand, mathematicians, and also some engineers, have legitimate concerns that a fundamental understanding of mathematics, as it applies to engineering, could be lost in these changes.

It is, in our opinion, especially insightful to try to understand how research carried out in other areas can contribute to the consideration of mathematical thinking and learning at post-secondary level. It can help us question theories and positions as regard mathematical learning and thinking that have generally been established having in mind more or less explicitly the mathematical education of "pure mathematicians" or mathematics teachers and their particular needs, and also that have focused on a few classical mathematical domains. It also obliges us to consider the role of technology in learning in a rather different way. Indeed, what is often at stake in these domains is not simply the use of technology for developing usual mathematical knowledge but the way in which technology is changing mathematical activity and understanding, including problem solving, proving, reasoning, modeling, and symbolizing. Mathematical and technological expertise and needs are in these domains much more tightly intertwined.

Learning Advanced Mathematics: The Case of Engineering Courses

This section provides a commentary on recent trends in mathematics education from the viewpoint of "service mathematics," that is, mathematics as taught to non-mathematics specialists and students studying science, engineering, and other technical subjects. We focus on trends in the United Kingdom and with respect to engineering students (Kent & Noss, 2003). We also consider the development of technology for teaching and learning in service mathematics.

In this section, we move away from a concern with what might be called intra-mathematical structure (and cognition), such as debates on the role of proof in mathematics, or advanced mathematical thinking as discussed earlier, and we will be more concerned with questions of *modeling*—the relationships between mathematics and other knowledge domains, and the ways in which computer technologies are shaping these relationships.

We begin with a sketch of the area of nonspecialist mathematics practice and research. There is in fact rather little tradition of educational research in the nonspecialist area, although arguably this area of research could bring many benefits, given that in terms of the numbers of students involved there are far more than the number of specialist mathematics students, and it is also the area of mathematics education with perhaps the most pressing need for changes in curriculum and teaching approaches, as the "customers" (engineering students and engineering academics) are by many accounts increasingly dissatisfied with the traditional curricula and approaches offered by the mathematicians who teach nonspecialist courses. Up to now, there have been very few studies of the different ways that mathematics and engineering students think about mathematics (see Bingolbali, 2004; Maull & Berry, 2000). One of these few has been Magajna's interesting and very detailed study of students and their mathematical thinking while studying at vocational college to become engineering technicians (Magajna, 2001; Magajna & Monaghan, 2002).

Innovation in engineering mathematics teaching tends to be practitioner-led, that is, it is carried out by university lecturers themselves, who (in most countries) divide their time between teaching and research. Educational research papers, if written at all, tend to take the form of descriptive reports, not much connected with the research literature of the world of mathematics education. Where a distinct research methodology is followed, the pre-test/intervention/posttest approach is still quite common, which is nowadays out of favor amongst socioculturally influenced mathematics educators. It remains very much an open question what kinds of research can particularly contribute to promoting change amongst higher education teachers (including connecting them to educational research), who have traditionally asserted a high degree of autonomy in their professional lives, including matters of teaching, and often harbor a somewhat negative attitude towards mathematics education (the "math wars" that broke out in the 1990s in the USA illustrate some of the tensions involved here—see Ralston, 2004). There have been some attempts to build a research culture for educational innovation in university teaching. In the UK, for example, the Higher Education Academy Subject Network is a government-funded initiative that aims to support lecturers in all major subject disciplines, including the branches of engineering and science, to carry out educational projects, share their results, and generally network ideas about learning and teaching; one of the significant features of the Network is that it is based on subject-specific support centres, directed and staffed mainly by lecturers from that subject area—the implication being that lecturers tend to feel closer in thinking to their fellow subject professionals than to educational specialists.[19]

The last 10 years have been a very interesting period of change for nonspecialist mathematics teaching—for a range of accounts of these developments see the outcomes of the ICMI Study on the Teaching and Learning of Mathematics at University Level, which took place in 1999 and was published in two volumes of papers (Holton, 2000, 2001); see also the proceedings of a regular conference on developments in engineering mathematics (Hibberd & Mustoe, 1997, 2000; Mustoe & Hibberd, 1995). After many decades of stable curricula and teaching methods—typically, substantial lecture courses of up to 100 hours per academic year delivered by mathematics lecturers to large groups of nonspecialist students (and the notion of delivery very much underlines the most common

[19] See, for example, the Mathematics, Statistics and OR Network [http://www.mathstore.ac.uk] and the Engineering Subject Centre [www.engsc.ac.uk].

approach, based on transmission of mathematical content, accompanied by students doing extensive exercises in techniques)—recent years have seen some major changes. The impetus for change in mathematics teaching has come from several directions. From the direction of professional engineering comes the enormous change throughout professional practice with the arrival of cheap and ubiquitous computer technology. In many cases (in civil engineering, for example), this has fundamentally changed the technical nature of work, and this is now being reflected in a deep-rooted re-appraisal of undergraduate engineering education. Indeed, new attitudes to "knowledge" are developing in engineering education (Kent & Noss, 2003). More and more knowledge is becoming accessible and potentially relevant to the practicing engineer—not only mathematics and physical science, but knowledge concerned with materials, construction techniques, design methodology, project finance, law, environmental issues, and so on. Broad agreement is emerging amongst engineers that the way to deal with this "knowledge explosion" is to implement a shift in emphasis from teaching built around subject knowledge (i.e., the topics of engineering theory and science, and issues of professional practice) toward teaching about the process of engineering (how knowledge is operationalized), using engineering design as an organizing and motivating principle of an engineering degree.

The other direction of impetus comes from school mathematics: During the 1990s, in the UK (where expansion of university-level education was and remains a key government education policy) and in some other countries, a growing proportion of young people went to university. At the same time, this was accompanied by a widely perceived (but usually officially denied) perception that the level and overall quality of mathematical preparation in schools has declined (Smith, 2004, reports on a major investigation of this phenomenon). To what extent the decline is true or not, certainly whereas in the past a relatively elite group of students would enter university to take degrees in engineering and science, today's students come from a far more diverse background and have a far broader range of mathematical competence (cf. SEFI, 2002). School mathematics in many countries has also changed radically during the last 20 or so years—a common significant change is the "democratization" of mathematics in schools, a movement that has seen a decline in formal mathematical activities by students (exercises in geometry, algebra, calculus, etc.—mathematics regarded as a body of knowledge to be acquired) and an increase in student-generated investigative activity (projects, etc.—mathematics regarded as a process in which students can participate).

From the perspective of service mathematics, university teachers ironically seem to be unhappy with a style of mathematics curriculum in schools that emphasizes a way of working that can help to develop students' abilities in problem solving and applying mathematical ideas—of course, this very much depends on how, and how well, this is done. Yet one can sense a missed opportunity in this situation: Where there seems to be a possibility for making connections, the debate in universities has focused on the emergence of a gap between the expectations of universities about new students and the students' actual mathematical capabilities; this has been termed the *mathematics problem* in the UK. To a significant extent, however, arguably the notion of "problem" is more a reflection of a dominant conservative viewpoint towards mathematics teaching than a problematic reality. As Steen (2001) describes, through the particular influence of the computer mathematics is becoming more and more relevant to more and more professional workers, including engineers, and therefore a rich *opportunity* is opening up for mathematics education, but this will require mathematicians to be more open to consider changes in the content and approach of mathematics teaching—either that, or to see more and more mathematics teaching being undertaken by the users of mathematics, such as engineering departments, themselves. Here, for example, is a comment recorded (by PK) from a mathematician specializing in engineering mathematics teaching at a large technical university in the UK: "We're pretty traditional—you would not see much difference between what we do now and 20 years ago, except that the level is lower now."

The Influence of Technology

Implementation of technology has been a big concern generally in university teaching since the early 1990s, when personal computers became accessible in significant numbers to students in higher education. One of the striking things about this has been the tendency for researchers and especially implementers to regard the higher education setting as distinct from the school setting, and many lessons learnt from the implementation of IT in school level education have noticeably *not* been learnt by those in higher education. For example, technology enthusiasm was particularly rife in the early to mid-1990s, much as in schools about a decade earlier, often in the form of expensive projects that generally proved unsustainable after the special project funding was taken away.

> The strength of enthusiasts is that they innovate, they try new ideas, they explore where more cautious colleagues might wait and see. Their weakness is that the outcomes of their enthusiastic endeavours are often poorly contextualised, theorised or evaluated. (Burton et al., 2004, p. 221)

It is pretty much recognized, by mathematics educators at least, that for technology to have a sustainable impact on teaching and curriculum, then there has to be some deep-rooted thinking about the aims and methods of teaching—as the anthropological approach in mathematics education illustrates (see above). Of course, to do this in the complex political atmosphere that surrounds a typical university mathematics programme is no easy task, and even today one is inclined to think, pessimistically, that "university mathematics curricula are virtually immune to change; in mathematical terms, we might say that the mathematics curriculum is invariant under intellectual revolutions" (Steen, 1989).

For all the difficulties inherent in technological innovation in mathematics education, it is evident that those involved in teaching mathematics to nonspecialist students cannot ignore it, inasmuch as computer software is ubiquitous in the professional practice of engineers and is becoming a standard tool for engineering students, for example in the form of specialist Computer-Aided Design packages for the different engineering professions, or structural analysis packages in civil engineering.

The case of structural analysis software is a good illustration of the trends. Structural analysis is essentially a deeply mathematical subject and is central to the professional practice of a civil engineer, and thus also central to his or her education. As part of the design process, the engineer needs to predict the behavior of a proposed structure (the walls of a building, a roof, a bridge, etc.). Before the advent of computers, the working life of an engineer, especially in the early part of his or her career, would be dominated by actually doing structural calculations using pen-and-paper, and a large part of the civil engineering degree was therefore dedicated to giving students an understanding and fluency in a variety of calculational techniques. For the majority of engineers today, all such calculations will be done in practice using computer software. The priority now for education is that students may "do" little mathematics themselves in the form of explicit calculation, yet they are likely to *use* more mathematics than ever before, implicit within computer software:

> No longer do we have to grind through long calculations—the computer will do it for us. The challenge has changed from the ability to do this to the ability to interpret the meaning of mathematics to engineering and herein lies the challenge and change of emphasis. (Blockley & Woodman, 2002, p. 14)

A typical structural analysis curriculum of 10 years ago would begin with a solid dose of analytical theory—using matrix algebra—followed later (perhaps in the 2nd or 3rd year) by working with computer software. Nowadays, work with computer software is likely to come much earlier, and the place of learning matrix algebra as a preparation for working with software is becoming open to question.

What is the Purpose of Mathematics to an Engineer?

Structural analysis is just one of the analytically based areas of engineering where the curriculum is currently being debated and is evolving into new forms (see, for example, Allen, 2000). In general, one can see a tension between the traditional notion of "pushing" mathematical ideas into the engineering curriculum and of "pulling" ideas from mathematics—that is, the need for a particular piece of mathematics can emerge where the engineering curriculum requires it, rather than being pushed into the student prior to having a meaningful context for it. Of course, this has to be done in a systematic way—mathematics cannot in general be understood in a "just in time" fashion.

In university mathematics education, a strong tendency remains to think that *understanding* of a mathematical technique *must* precede its *application* to the engineering context. In the (precomputer) past, a valid objection to any sort of pull-based mathematics was the uncomfortable notion of anyone trying to use a mathematical idea before knowing the techniques of its application. Mathematicians have tended to regard knowledge of techniques as an essential part of what it means to understand an idea and how to apply it. What users of mathematics such as engineers are wanting however, and it seems that mathematicians will increasingly need to provide, is a form of pull-based mathematics in which the use of mathematical software makes mathematical ideas usable, as in the example of structural analysis above. Carefully designed use of IT can make it possible for students to use mathematical ideas *before* understanding techniques, and to make this part of a genuinely rounded mathematical learning experience (a few examples are given in Kent & Noss, 2003).

It is worth considering at this point the whole question of the purpose of mathematics to a (student) engineer. Kent & Noss (2003) in a UK-based study found widespread agreement across the civil engineering profession about what is the desirable mathematical competence of a graduate: "Mathematics should instil disciplined thinking and rigour in the development of arguments based on assumption and simplification in modelling, should teach the importance of controlled approximation, and, above all, impress upon students its value as a tool to be invoked when quantitative evidence is needed to underpin assertion, hypothesis, or sheer physical intuition" (Nethercot & Lloyd-Smith, quoted in Kent & Noss, 2003, p. 8). Whilst few would dissent from such a description, there are certainly significant questions about what meanings people attach to the different terms, even, what is "mathematics"? It is notable in talking to experienced engineers that the mathematics that is useful and relevant has in many cases long since come to be thought of as part of engineering, whereas mathematics that is *not* used is regarded as "mathematics."

Again according to Kent and Noss's (2003) survey, practicing engineers do not tend to regard mathematics as a problem area—contrary to the sometimes obsessive concern of academics about the "mathematics problem" (see above). Practitioners regard confidence with mathematics as crucial for the majority of engineers, and above all they require a balance of skills across engineering teams. Different employers require different balances, and even specialist analytical design consultancies in civil engineering reported that they only require 10–20% of their engineers to have specialist skills in analysis.

In the past, engineers had to learn a lot of mathematics for practical purposes. At the same time, they could be expected to absorb some understanding of mathematics as a logical way of thinking, and its importance as part of a practicing engineer's expertise. The availability of computer software for calculations has undone this relationship between practical and theoretical aspects. The teaching of practical mathematics is becoming much more focused on the process of modeling of engineering systems—this results in a decrease in the teaching of calculation techniques, but it does not mean that all manual work with mathematics could be replaced: The right balance must be found. To teach mathematics as a way of thinking, a traditional model exists in the use of formal mathematics—older engineers do look back on courses in Euclidean geometry in their schooldays as having significantly shaped their development. It seems unlikely that formal geometry can re-occupy such a dominant place in the school mathematics curriculum (for one thing, there are too many competing demands on the school curriculum), and new models for this aspect of mathematics learning are not yet widely agreed. Indeed, is a logical way of thinking *only* to be gained through studying "pure" mathematics? Could the same kinds of analytical problem solving also be developed through, for example, requiring all students to develop substantial skills in (mathematical) computer programming? Certainly in the school context this potential for computer programming has long been advocated, most famously in the case of the language Logo (see Cuoco et al., 1996; Noss & Hoyles, 1996).

Modeling and the Development of "Problem Solving" Curricula

Another emergent factor in engineering mathematics is the role of modeling. However, this should be understood more broadly than it is sometimes interpreted in undergraduate mathematics. Often, the translation between the physical situation and the mathematical analysis is regarded as a small part of the task, whose focus is on the applied mathematics of solving the equations in the model (in the worst cases, the physical situation is but an excuse for students to do a routine exercise in applied mathematics). Yet the process of translation, which involves on the one hand developing a mathematical model of the situation and on the other hand interpreting the mathematical analysis back into the context, is in general a rather complex task, and is intimately related to the issue of learning *how to use* (rather than how to do) mathematics, which we suggested above is becoming the focus of engineering mathematics education. See Bissell and Dillon (2000) for an extremely perceptive engineers' view on the nature of modeling.

As discussed above, an analogous situation to this lies in some aspects of applied mathematics teaching, such as differential equations (see, for example, Stephan & Rasmussen, 2002). As the use of differential equations has been transformed by the availability of mathematical software, both numerical and symbolic (computer algebra systems), both a possibility and a need has opened up to teach about equations differently, less focused on doing techniques of solving differential equations, more focused on the meaning of differential equations, in terms of their geometrical structures and their domains of application.

One of the major trends of recent years in engineering education generally that illustrates the issue of modeling is a growing number of experiments with various forms of *problem-based learning* (PBL). (A few exceptional universities—such as Roskilde University in Denmark—embraced this mode of learning a long time ago, cf. Niss, 2001). The idea behind this is that lecture-based teaching is largely replaced by students working on projects, usually in teams, and being assessed through group-work and continuous assessment. The projects develop increasing complexity as students progress from year to year. Problem-based learning originated in the 1960s in medical education, where it has since become increasingly common. The appropriateness of PBL has been criticized for engineering studies because of its origins in medical education: The argument goes that PBL is good for fact-based knowledge, such as medicine, but is less appropriate for an analytically based subject like engineering (cf. Spencer-Chapman, 2000). Yet making abstract analysis meaningful depends crucially on the student making connections between engineering and mathematics through solving problems (Kent & Noss, 2003); just as in the case of differential equations mentioned above, meaning develops by providing activities for students to develop connections within the subject (e.g., the geometrical structure of differential equations), as well as making connections to other subjects.

Statistics and Probability at Post-Secondary Level

In this section we summarize the research carried out in advanced stochastics, with particular emphasis in the recent emergence of a statistics education research community, which has some links but is almost independent from the advanced mathematics education research community. We reflect on the specific characteristics of advanced stochastics and the potential research challenges that this area sets to educational research.

Many features described for the teaching of "service mathematics" to engineers also apply to the case of statistics. As mentioned above, statistics is one of the most widely taught topics at university level, where it is mainly studied as a tool to solve problems in other fields. However, the variety of previous knowledge and interests of the students involved is even larger than in the case of service calculus, because statistics is taught with almost no exception in all the university majors, from engineering to education, geography, psychology, biology, sociology, journalism, economics, linguistics, and so on. There is also a general dissatisfaction with the mathematically oriented approach to teaching in traditional courses, which often emphasize the teaching of formulas for calculating statistics (e.g., correlation coefficients or confidence intervals) without much concern towards the data context and interpretative activities or simulations that could help students improve their stochastic intuitions. In many cases, the courses are over-mathematized, which involves meeting concepts of advanced stochastic thinking for students who do not have any prior or concurrent experience of advanced algebra or calculus.

Technology has also had a major impact in the practice and development of statistics in expanding the range of processes that statisticians and users of statistics can employ to collect, analyze, and interpret data and the amount and type of data they can analyze. This has resulted in the development of new statistical methods, such as exploratory or graphical data analysis, resampling techniques, or data mining and has facilitated the implementation of some techniques that were previously difficult to make use of, such as multivariate or Bayesian analysis. This no doubt has also influenced the increasing demand for statistics education, inasmuch as many people can now apply a method with scarce knowledge of the complex mathematical calculation behind the user-friendly software that is apparently easy to learn. For example, an education or psychology student can easily enter data in a computer, use statistical software to perform a factor analysis in a few minutes, and give an intuitive interpretation of the factors retained in terms of the problem without knowing what an eigenvalue is, how eigenvalues are extracted, or how the different rotations of the components have been obtained.

Modeling and problem solving are also central to the teaching of statistics to professionals. However, statistical problems are often open-ended or ill-defined and have multiple possible solutions. Although model abstraction is common to mathematics and statistics, the context plays a fundamental role in guiding the selection of a statistical model (delMas, 2004), and selecting the model is frequently harder than subsequent mathematical reasoning in working with the model. Algebraic work on the model, for example, is very limited, and although the processes of generalizing, classifying, conjecturing, inducing, analyzing, synthesizing, and abstracting described by Dreyfus (1991) for advanced mathematics still apply to statistics, the meaning of representing, visualizing, or formalizing is quite different. As we will argue later, representing and visualizing apply mainly to the data and are used to create meaning from these data, for example helping to find an adequate model or identifying unexpected sources of variation. Formalizing in statistics applies not just to the mathematical work with models, but to the definitions of units, variables, and categories for data analysis, or designing instruments to measure them, which again are rooted in the context of application. Whereas mathematical practice can be removed from real-world context, statistics practice is highly dependent on the problem context, and this dependence may lead to reasoning errors that are hard to overcome.[20]

There is an interest and tradition for educational innovation in the teaching of statistics, and this topic is also taught at advanced level by lecturers with a variety of backgrounds, mostly statisticians, but also economists, health care professionals, engineers, psychologists, or educators. Contrary to the case of engineering, mathematicians or mathematics educators teaching statistics are only a minority. This explains the fact that, until very recently, research in advanced stochastic teaching and learning has not attracted mathematics educators and thus has had a small presence at, for example, the annual Psychology of Mathematics Education Conferences. Of course, this does not mean that the teaching of advanced statistics is without problems or that existing related research on this theme is lacking.

A main difference compared with the teaching of mathematics to engineers is the long tradition of educational research related to the teaching and learning of statistics, and a substantial part of it has been carried out outside the mathematics education community. Research in stochastics thinking, teaching, and learning started during the 1950s with the pioneering work by Piaget and Inhelder (1951) and has always had an interdisciplinary character. As stated by Shaughnessy (1992, p. 465): *"The cross fertilization of research traditions and methodologies in probability and statistics makes it one of the theoretically richest branches in mathematics education."* Different fields have contributed with various research paradigms and theoretical frameworks, which have been analyzed in different surveys of research, such as Scholz (1991), Shaughnessy (1992), Jones and Thornton (2005), and other chapters in this *Handbook*. Whilst research was still relatively scarce when the first *Handbook* was published, compared with other areas of mathematics it has experienced a notable growth in the past 10 years. However, this has mainly concentrated in elementary and secondary school levels, perhaps unsurprising given the much greater emphasis that recent new curricula for these levels have given to statistics and probability.

Research into advanced stochastic reasoning has interested psychologists for decades. Because psychology is an experimental science that heavily relies on statistics, the efforts to justify the scientific character of this field have led psychologists to examine the validity of their research paradigms, including the use of statistics in empirical research. An amazing observation is that statistical inference and particularly significance tests were found to be misunderstood and misused by psychologists and experimental researchers at large over 30 years ago, and that the situation still persists in spite of strong debates ever since (Harlow, Mulaik, & Steiger, 1997; Morrison & Henkel, 1970). Moreover, researchers in the field of reasoning under uncertainty have suggested that, even after statistical instruction, students and professionals tend to continue to make erroneous stochastic judgements and decisions (Kahneman, Slovic, & Tversky, 1982). The diffusion of these research results and the increasingly easy access to powerful and user-friendly computers and statistical software, which save teaching time previously devoted to laborious calculations and allow a more intuitive approach to statistics (more real data, active learning, problem solving, and use of technology to illustrate abstract concepts through simulation) have led statistics lecturers to increase their concern towards didactical problems (see, e.g., Moore, 1997). Consequently, a general effort exists to create curricular materials and to evaluate teaching and learning at university level. The influence of the International Association for Statistical Education (IASE) created in 1991 serves to establish links among the different communities interested in statistics education and to support a more systematic research program. Below we summarize these contributions.

Psychological Research on Advanced Stochastic Thinking

Different theoretical approaches in the field of psychology have tried to explain people's poor performance in probability and statistical tasks, which have been widely documented in relation to different concepts that can be considered as part of advanced stochastics, such as randomness, compound probability, association in contingency tables, correlation, conditional probabilities, Bayes problems, sampling, and the test of hypotheses. Traditionally, decisions under uncertainty are defined by incomplete information about the situation, that is, the possible alternatives or their outcomes are only known (at best) in term of probabilities, a feature of many professional tasks (e.g., medical diagnosis, jury verdict, educational assessment). A *fallacy* is the result of a cognitive process that leads to an incorrect conclusion and may consist

[20] For example, a very simple rule is computing the probability of the conjunction of two events in terms of simple and conditional probabilities: $P(A \cap B) = P(A) \cdot P(B|A) = P(B) \cdot P(A|B)$. A consequence of this rule is that the probability $P(A \cap B)$ cannot be higher than the probabilities of either of the two individual events. However, when $P(A)$ is very high and $P(B)$ small, people forget the rule and consider $P(A \cap B) > P(B)$. This behavior was first described by Tversky and Kahneman (1983) and termed the *conjunction fallacy*. So, for example students who are asked whether it is more likely that the government will increase the number of grants or that the government will increase the number of grants and the salary of members of parliament will consider more likely the second event.

of the application of an inadequate model or incorrect application of intuitive rules of inference. In trying to explain the mechanisms leading to a fallacy, Kahneman and his collaborators developed the *heuristics and biases* program (Kahneman, Slovic, & Tversky, 1982), which was the dominant paradigm in the early eighties and is still very influential. This assumes that people do not follow the normative mathematical rules that guide formal scientific inference when they make a decision under uncertainty and that, instead, they use simpler judgmental heuristics. Heuristics reduce the complex tasks of assessing probabilities and predicting values to simpler judgmental operations and are in general useful; however they sometimes cause serious and systematic errors and are resistant to change. For example in the *representativeness* heuristics, people tend to estimate the likelihood for an event on the basis of how well it represents some aspects of the parent population. An associated fallacy that has been termed *belief in the law of small numbers* is the belief that even small samples should exactly reflect all the characteristics in the population distribution. Then for example a mother of two boys may incorrectly assume that the likelihood of having a third baby who is a girl is greater than the likelihood of having a baby boy, because a family with 2 boys and a girl is more representative of the general population.

A huge amount of research in stochastics education at both undergraduate and advanced level has been based on the idea of heuristics (see a summary in Shaughnessy, 1992). Large-scale studies carried out by psychologists showed misconceptions related to these heuristics, in which participants assumed that sampling distributions (distribution of the value of the mean and other parameters in repeated samples from the same population) were independent of the sample size, whereas in mathematical theory the spread of these distributions should be inversely related to the sample size. Direct consequences of this for the practice of statistics are that users have an overconfidence in the results of statistical tests, underestimate the width of confidence intervals, and expect a very close result in replication of the experiment, even with a small sample. Recent research on heuristics at advanced level includes studies on: assessment (Garfield, 2003; Hirsch & O'Donnell, 2001), the effect of teaching experiments on the use of heuristics (e.g., Barragués, 2002; Pfannkuch & Brown, 1996), and misconceptions related to concepts such as randomness (Falk & Konold, 1997) in terms of heuristics or providing alternative explanations for incorrect responses to tasks proposed in research about heuristics. For example, Konold (1991) suggested that students might confuse frequentist probability, which is objective and refers to the frequency of occurrence of an event in relation to a population, with epistemic (subjective) probability referred only to an isolated event. Then they might confuse the task of assigning a frequentist probability with the task of predicting the next outcome (the *outcome approach*) in a random experiment. This also involves a deterministic view of an uncertain situation, because in the outcome approach, participants tend to rely more on causal explanations rather than accept explanations due to chance and variability.

A different theoretical model in the psychology of decision is the *abstract–rules* framework (Nisbett & Ross, 1980) in which people are assumed to acquire a form of correct statistical reasoning and develop intuitive versions of abstract statistical rules, such as the law of large numbers, which are well adapted to deal with a wide range of problems in everyday life, for example, estimating the average time spent to perform a repetitive task or the approximate cost of the shopping in a supermarket, but the rules fail when applied beyond that range. Such rules are used to solve statistical problems, in which one recognizes some cues in the problematic situation. With respect to training in statistical reasoning the suggestion is that the quality of human inferences and judgments can be improved via statistical instruction: "Many of the inferential principles central to the education we are proposing can be appreciated fully only if one has been exposed to some elementary statistics and probability theory" (Nisbett & Ross, 1980, p. 281), although the suggestion is that statistics teaching should take into account students' intuitive strategies and errors. Moreover, for these authors, even brief formal training in inferential rules may enhance their use for reasoning about chance events; either teaching statistical rules or teaching by having students solve example problems would work. As regards training in conditional logic or conditional probability students need to be trained simultaneously in abstract logical rules and in problem-solving strategies. Problem solving is improved, according to these authors, when the sample space is clearly defined, people recognize the role of chance in the experiment, and the context forces the person to think statistically.

A more recent theoretical framework is the *adaptive algorithms* approach (Cosmides & Tooby, 1996; Gigerenzer, 1994), which supposes that people possess evolutionarily acquired cognitive algorithms that serve to solve complex probability problems, such as conditional probability or problems involving Bayes theorem. Adaptive algorithms serve to solve adaptive problems (such as finding food, avoiding predation, or communicating) and take a long time to be shaped, due to natural selection. Because adaptive algorithms are shaped by natural environments they are more effective when the tasks are presented in a format close to how data are perceived and remembered in ordinary life. According to this theory people should have little difficulty in solving statistical tasks if the probability data are presented in a natural format of absolute frequencies instead of using rates or percentages (Sedlmeier, 1999). Although the percentage, rate, or frequency representations of probability values in the problem are mathematically equivalent, inasmuch as they can be mapped onto each other, they might be not psychologically equivalent. Gigerenzer (1996) argued that the frequency representation makes some cognitive illusions in statistics disappear and that the cognitive algorithm used to solve the same problem changes with the information representation.[21]

Sedlmeier (1999) analyzed and summarized recent teaching experiments carried out by psychologists that follow either the abstract-rules or adaptive-algorithms theories and involve the use of computers. The results of these experiments favor the adaptive-algorithms approach as an explanation for people's performance in stochastic tasks and suggest that statistical training is effective if students are taught to translate statistical tasks to an adequate format, including tree diagrams and absolute frequencies. These experiments have concerned areas such as compound probabilities, conditional probability, the Bayes theorem, and the impact of sample size on the sampling distributions. However, learning is assessed through participants' performance to tasks that are very close to those used in the training, so evaluating the extent to which students could transfer this knowledge to wider types of problems is difficult. Although this research provides empirical information and potential theoretical explanations for students' difficulties in advanced stochastics, a large amount of work still must be done by mathematics and statistics educators to integrate these results and to use them to design and evaluate teaching sequences in natural settings where students are expected to meet wider curricular requirements.

The Mathematics Education Approach

The approach of mathematics educators is rather different from that taken by psychologists and has followed the research paradigms and theoretical frameworks in mathematics education. The mathematical and epistemological analyses reveal that the complexity of concepts, tasks, and students' responses investigated by psychologists is often greater than has been assumed in psychological research and suggests the need for reanalyzing them from a mathematical perspective; for example, centering on isolated types of tasks does not always reveal in depth the students' understanding of a concept, inasmuch as responses are sometimes very dependent on the task variables. Moreover, theoretical constructs taken from mathematics education can contribute to a different perspective for the same phenomena. Take, for example, the concept of correlation, a field that has been extensively studied by psychologists, including Inhelder and Piaget (1955) who considered that the evolutionary development of the concepts of correlation and probability are related and that understanding correlation requires prior comprehension of proportionality, probability, and combinatorics. In these experiments participants are presented two-way tables (two rows and two columns) in which a sample subjects are classified according to the presence-absence of two qualitative attributes, such as eye and hair color (dark or fair hair, brown or blue eyes). Participants are asked to judge if there is a relationship between the two attributes (judgment of correlation) in these 2×2 contingency tables and to justify the procedure to reach their conclusion. In Table 22.1 we describe the data in this type of problem, where a, b, c and d represent absolute frequencies.

Table 22.1 Typical Format for a 2×2 Contingency Table

	B	Not B	Total
A	a	b	a + b
Not A	c	d	c + d
Total	a + c	b + d	a + b + c + d

Piaget and Inhelder found that some adolescents who are able to compute single probabilities only ana-

[21] For example, in the following Problem 1 (Eddy, 1982) probability data are given in percentages and should be solved with the help of Bayes theorem. Giving the data in a format of absolute frequencies (Problem 2), it transforms to a simpler problem that can be solved by applying the Laplace rule: It is easy to see that out of 107 women with positive test results (8 + 99) only 8 of them will have breast cancer.

Problem 1. The probability of breast cancer is 1% for a woman at age 40 who participates in routine screening. If a woman has breast cancer, the probability is 80% that she will have a positive mammogram. If a woman does not have breast cancer, the probability is 10% that she will still have a positive mammogram. Imagine a woman from this age group with a positive mammogram. What is the probability that she actually has breast cancer?

Problem 2. Ten out of every 1,000 women at age 40 who participate in routine screening have breast cancer. Of these 10 women with breast cancer, 8 will have a positive mammogram. Of the remaining 990 women without breast cancer, 99 will still have a positive mammogram. Imagine a woman from this age group with a positive mammogram. What is the probability that she actually has breast cancer?

lyze the cases in cell [a] in Table 22.1 (presence-presence of the two characters A and B). When they admit that the cases in cell [d] (absence-absence) are also related to the existence of correlation, they do not understand that cells [a] and [d] have the same meaning concerning the association, and they only compare [a] with [b] or [c] with [d] instead. Understanding correlation requires considering quantities (a + d) as favorable to the association and (b + c) as opposed to it. The correct strategy should use the comparison of two probabilities *P(B/A)* and *P(B/NotA)*, that is

$$\frac{a}{a+b} \text{ with } \frac{c}{c+d}.$$

According to Piaget and Inhelder, recognition of this fact only happens at 15 years of age. Following Piaget and Inhelder, several psychologists have studied the judgment of association in 2 × 2 contingency tables in adults, using various kinds of tasks. Other psychologists used instead two numerical variables and different numerical, verbal, or graphical representation of data. As a consequence of all this research, it has been noted that participants perform poorly when establishing correlation (see Beyth-Marom, 1982, for a survey) and people frequently use their previous theories about the context of a problem when judging association. The general conclusion is that when data do not coincide with these expectations a cognitive conflict affects accuracy in the perception of correlation (Jennings, Amabile, & Ross, 1982).

Estepa and his collaborators began from these results and, after a mathematical and epistemological analysis of the concept of correlation, designed a questionnaire including a wider range of open-response tasks to give an alternative explanation of pre-university students' strategies and judgments in terms of *misconceptions* regarding correlation (Batanero, Estepa, Godino, & Green, 1996; Estepa, 1993). In the *determinist* conception of association, some students expect that correlated variables should be linked by a mathematical function; students with a *unidirectional* conception of association only perceive direct association (positive sign of the correlation coefficient) and interpret inverse association (negative sign) as independence; *causal* conception consists of assuming that correlation always involves a cause-and-effect relationship between the variables, and in the *local* conception students base their judgment on only part of the data (e.g., they only use one cell in 2 × 2 contingency tables). These types of conceptions have been confirmed in later research (Morris, 1997, 1999). Another finding was that some mathematical concepts might constitute an *obstacle* when learning correlation, in which case they receive different interpretations. For example, some students believe a correlation coefficient of –0.7 indicates a smaller degree of dependence than a correlation coefficient of 0.1 because –0.7 < 0.1 in the usual ordering of real numbers, whereas in fact –0.7 indicates a stronger dependency than 0.1.

Estepa and his collaborators organized two teaching experiments based on *computer learning environments* (in the sense of Biehler, 1994, 1997), that is, integrated instructional settings that allow the teacher and a group of students to work with statistical software, data sets, and related problems as well as with a selection of statistical concepts and procedures. The researchers found general improvement in students' strategies and conceptions after teaching, although the unidirectional and causal misconceptions concerning statistical association were harder to eradicate. Using qualitative methods, such as observation, interviews, and the analysis of the students' interaction with the computer, they documented specific *acts of understanding* (following Sierpinska, 1994) of correlation (Batanero, Godino, & Estepa, 1998; Estepa, 1993). For example, students must realize that *the study of the association between two variables has to be made in terms of relative frequencies;* however, in the first teaching session the students tried to solve the problems in terms of absolute frequencies. Although the lecturer pointed out this mistake to them at the end of that session, the same incorrect procedure appeared recurrently for the same students for several sessions, until the students overcame this difficulty. Another example of an act of understanding was realizing that *from the same absolute frequency in a contingency table cell one can compute two different relative conditional frequencies (conditioning by row or column), and the role of the two events in the conditional relative frequency is not interchangeable.* Students' difficulties in discriminating between the probabilities P(A/B) and P(B/A) have previously been described (e.g., Falk, 1986). Again, this was a recurrent difficulty for the students throughout the teaching experiment.

Other researchers have taken mathematics education frameworks to organize research on topics that have scarcely attracted psychologists in complex experimental settings. For example Batanero, Tauber, and Sánchez (2004) used a theoretical framework that incorporates ontological, anthropological, and semiotic ideas (Godino, 2002; Godino & Batanero, 1998) to describe the evolution of students' understanding of the normal distribution in a statistics course based on intensive use of computers. In the theoretical model used, the meaning (understanding) of any mathematical object is conceived as a complex system, that contains different types of interrelated elements: a) problems and situations from which the object emerges, such as fitting a curve to a histogram for empirical data distributions or approximating the binomial or Poisson distributions; b) representations of data and concepts, for example, histograms or density curves, verbal and algebraic representations of the normal distribution; c) procedures and strategies to solve the problem, such as computing probabilities under the curve, computing standard scores, and critical values; d) definitions and properties, for example, symmetry of the normal distribution, horizontal asymptote, areas above and below the mean, meanings of parameters; and e) arguments and proofs, including deductive and informal arguments. Godino and Batanero (1998) argued that the understanding of a concept emerges from a person's meaningful practices linked to repeated solution of problems that are specific to that concept. Mathematical problems promote and contextualize mathematical activity and, together with actions, constitute the praxemic or phenomenological component of mathematics (*praxis*) as proposed by Chevallard (1997). The three remaining components (concept-definitions, properties, arguments) are produced by reflective practice and constitute the theoretical or discursive component (*logos*). Godino and Batanero also described different dual facets or dimensions of mathematical knowledge, in particular distinguishing between the *personal* and the *institutional* meaning to differentiate between the meaning that for a given concept has been proposed or fixed in a specific institution and the meaning given to the concept by a particular person in the institution.

This theoretical framework was specially suited to describe the complexity of understanding the normal distribution in Batanero, Tauber, and Sánchez's (2004) research, because the understanding of the normal distribution in their experiment was based on the students' previous knowledge about many interrelated concepts such as statistic and random variables, frequency and probability distribution, parameter and statistics, center and spread, symmetry and kurtosis, histogram and density curve, areas under the normal curve, mathematical model and empirical data, their different representations, procedures, and properties. The teaching was based on intensive solving of real problems in which students analyzed different real data sets taken from fields in their areas of interest, which served to gradually suggest to them to reflect about the different elements of meaning of the normal distribution. The idea of different institutional meanings also describes well the changes in meaning and understanding implied by the use of computers: different types of problems (starting from real, multivariate data sets related to a problematic situation that the students should model), representations (wider and quicker availability of interactive dynamic graphs, tables, and numerical summaries), procedures (knowledge of software options and interpretative abilities substituted for classical probability computations); graphical and iconic language replacing algebraic manipulation, and types of proof (simulation and visualization, instead of deductive proof).

Finally the framework served to picture the general tendencies in the students' *personal meaning* for the normal distribution after the instruction, as well as the variety of these meanings, and to identify main agreements and differences with the intended *institutional meaning*. At the end of the teaching students were given written questionnaires to evaluate specific understanding about particular properties, representations, and procedures and an open-ended task, to be solved with the computer that referred to a data file the students had not seen before. The students were to chose a variable that could be well fitted by a normal distribution, then explain and justify their responses in detail (the file contained data from a real context with 10 different qualitative and quantitative variables, only 2 of which were acceptable solutions to the problem). The students were given freedom to use any resource of the software (statistical graphs or analysis) to support their choice. Quantitative analysis of responses to questionnaires and semiotic analysis of the students' written protocols in the open tasks as well as interviews with a small number of students were used to describe the students' correct and incorrect reasoning about the normal distribution. For example, many students could relate the idea of symmetry to skewness coefficient or to relative position of mean, median, and mode; they were able to compare the empirical histogram and density curve shapes to the theoretical pattern in a normal curve and relate the various graphic representations and data summaries to the geometrical properties of normal distribution. Even when the majority of students learnt to use the software, some of them had difficulties in interpreting areas in frequency histograms produced by the software, in computing areas under the normal curve, or in discriminating between the empirical data and the mathematical model (normal curve).

Influence from Statistics

Another strong influence on research comes from the field of statistics, where interest in education arose especially since the creation in 1949 of an Education Committee by the International Statistical Institute (ISI), through which the ISI promotes the training of official statisticians in developing countries and has organized a series of Round Table Conferences on specific educational problems since 1973 (Vere-Jones, 1997). The International Conferences on Teaching

Statistics (ICOTS) were started in 1982 by the ISI and have continued every 4 years. In 1991 the IASE was created as a separate section of the ISI and took over the organization of these conferences. The journals *Teaching Statistics*, first published in 1979, and *Journal of Statistics Education*, started in 1993, soon became important tools to improve statistics education all over the world. This activity complemented the educational sessions at international and national statistical conferences (e.g., in the biannual Sessions of the International Statistical Institute or in the American Statistical Education Conferences) as well as the Stochastic Thinking Reasoning and Literacy Research Forum started in 1999. With regard to the teaching, learning, and understanding of stochastics at post-secondary level these conference activities promoted a large and varied number of teaching experiences, and many suggestions were published in proceedings and journals. At the same time, some systematic research has been progressively emerging under the influence of the IASE and of several funded projects and initiatives by the American Statistical Association (e.g., the Undergraduate Statistics Education Initiative, USEI,[22] and the Consortium for the Advancement of Undergraduate Statistics Education, CAUSE[23]).

One example is the research carried out by Garfield, del Mas, and Chance who developed a piece of didactic software (*Sampling Distribution*) and complementary instructional materials (pretest and posttest, instructional objectives, activities) to assess students' previous misconceptions, support discovery and exploration of inferential concepts, and assess change after instruction (Garfield, del Mas, & Chance, 1997). The software allows student to choose among a variety of possible distributions for a continuous variable in a population, including nonstandard models of distributions. In addition it provides different windows where the student can simulate the drawing of samples (for any size and number of samples) from the specified population and visualize the sample outcomes as well as the values of some statistics (e.g., mean, median, variance) in the different samples (sample statistics). From these, there is the possibility to graph the set of different values for these sample statistics to get an empirical sampling distribution for the statistics. Students can add new samples to their previous results little by little, in order to discover the long-run patterns in the empirical sampling distribution and understand the features in the theoretical sampling distribution (distribution of all the possible values of the statistics for a given population and sample size). For example, they can "discover" for themselves the central limit theorem, according to which the sampling distribution of mean and other statistics will approach to a normal distribution for moderate sample sizes ($n > 30$) even for nonsymmetrical population distributions. They can see how the approximation improves with the sample size, and that as the spread in the sampling distribution decreases the average (value of the sample statistics) approaches to the value of the population parameter.

The same authors carried out a teaching experiment with three groups of students at the university introductory level (delMas, Garfield, & Chance, 1999). The students were supposed to have read a chapter on sampling distribution and the central limit theorem before engaging with hands-on simulation using the software. Class discussions served to compare the shape, center, and spread of empirical sampling distributions for different sample sizes and population distributions and whether empirical results agreed with what was expected in the central limit theorem. The authors were surprised that after the first series of experiments many of their students still showed serious misconceptions. For example, the variability the students expected in the distribution was not consistent with the sample size, and they did not understand that the sampling distribution would resemble a normal distribution with increasing sample sizes. Much better results were obtained when the authors followed a model of learning through conceptual change in which students were asked to test their predictions and confront their misconceptions, following a constructivist model of learning. The authors here concluded that students with misconceptions have to experience contradictory evidence and reflect on this contradiction with their previous expectations before they change their views about random phenomena.

To gather more detailed information about students' reasoning they carried out several guided interviews that suggested a model of developmental stages in students' statistical reasoning: idiosyncratic, verbal, transitional, procedural, and integrated process reasoning (Chance, delMas, & Garfield, 2004), based on work by neo-Piagetian cognitive researchers (e.g., Biggs & Collis, 1991) who have examined the process of cognitive development in everyday and school contexts. In this model students progress from knowing words and symbols without understanding their meaning (idiosyncratic level) to a substantial understanding of the process of sampling and sampling distributions (process level) in a series of stages. Similar mod-

[22] http://www.amstat.org/education/index.cfm?fuseaction=usei
[23] http://www.causeweb.org/

els of developmental stages have also been proposed by some mathematics educators for some elementary stochastics concepts, in research carried out with primary and secondary school students (Jones, Langrall, Mooney, & Thornton, 2004). Finally, we should note this kind of research focusing on the identification on misconceptions and developmental stages is rather classical from a mathematics education point of view.

Although many different theoretical concepts in research carried out by statisticians are taken from psychology or mathematics education, some statisticians are trying to develop specific frameworks to describe statistical thinking as a specific type of thinking that recognizes the variation around us and includes a series of interconnected processes, aimed at identifying, analyzing, quantifying, controlling, and reducing this variation in order to improve or inform decisions and actions in many different fields (Snee, 1990). A notable example is the theoretical framework developed by Wild and Pfannkuch (1999) and Pfannkuch and Wild (2004), who conducted qualitative research on the activities carried out by students and professionals when engaged in statistical investigations from which the authors developed a complex four-dimensional framework including four components. The first component is termed the *statistical investigation cycle PPDAC (Problem, Planning, Data, Analysis, Conclusion)* and describes the activities carried out when identifying and solving a statistical problem that is embedded in a wider contextual problem. This cycle includes identifying and setting of the problem, planning for its solution, collecting data, data analysis, and reaching conclusions. Secondly there is an *interrogative cycle*, which is a generic process in statistical problem solving consisting in generating possibilities for causes or explanations, seeking or recalling information, interpreting, translating or comparing information, criticizing this information from both internal and external points of view, and judging the reliability or usefulness of the information. The third component in the model describes the *types of thinking* present in statistical problem solving. In addition to strategic thinking the authors define fundamental modes of stochastic thinking, which include recognizing the need for data, transnumeration (numerical transformation to facilitate understanding, for example, classifying, coding, or representing data), identifying, explaining controlling and measuring variation, use of mathematical models, and synthesis of statistical and contextual knowledge. Finally the model describes a series of *dispositions*, such as curiosity, imagination, or skepticism. Following the publication of the Wild and Pfannkuch paper a number of research projects have focused on teaching or assessing statistical reasoning and its several components as a whole. For example Pfannkuch, Rubick, and Yoon (2002) described how students use transnumeration or perception of variation to build up recognition and understanding of relationships of variables in small data sets. Another example is the research by Biehler (2005) who analyzed about 60 students' statistical projects and used Wild and Pfannkuch's model to develop an assessment scheme and a project guide to improve the quality of these projects.

The Challenges of Advanced Stochastics

The above summary of research suggests that probability and statistics pose a number of important challenges to research on mathematics thinking and learning at post-secondary level. First, the existing research has been carried out by different scientific communities, and not just by mathematics educators, and for this reason the sources of information are widespread and not always easily available. At the same time, the diversity of research problems and approaches is very wide, something that is also common for undergraduate stochastics. Some research agendas posed by Shaughnessy (1992) and Shaughnessy, Garfield, and Greer (1996), as well as the creation of the *Statistics Education Research Journal* with the specific purpose to promote research show a tendency to unification and linkage of the isolated pieces of research towards developing a more general knowledge about statistical education. However, the influence from philosophical views about randomness, probability, and statistical inference that still continue today is reflected in stochastics teaching and research: *"in the field of probability there continues an ongoing fierce debate on the foundations, even though for pragmatic reasons only, this debate has calmed down in recent times"* (Borovcnik & Peard, 1996, p. 239). These different views influence the role given to probability in the curriculum, from being the central core, to trying to teach statistics without resort to probability (focus on exploratory data analysis only) or favoring classical, Bayesian, or mathematical-abstract approaches to inference.

Secondly, in the case of statistics, the distinction between advanced and elementary topics remains very fuzzy, inasmuch as current curricula in many countries include ideas about association and inference in secondary school alongside situations that require mature stochastic thinking for correct interpretation, such as voting, investment, research planning, or quality control, topics that increasingly form part of the information to which many citizens and professionals are exposed. So, whereas advanced mathematical thinking tends to be used only formally and after some systematic training, advanced stochastic thinking is now being formally or informally used by many people with little formal training in either mathematics or stochastics. Moreover some apparently simple concepts, such as randomness, that are taken for granted in elementary courses are in fact very complex. For example, judging whether a sequence of outcomes is likely to have been randomly generated involves being able to measure a number of parameters including relative frequency, length and distribution of runs, variation found in subsets, estimating probabilities, and so on. These skills are frequently taught in school; however research in both psychology and education has shown frequent errors and misconceptions related to such simple ideas as probability, average, and distribution. Furthermore, because no random sequence of outcomes exactly fits the expected patterns suggested by probability theory, judging randomness in a particular situation also requires an understanding of hypothesis-testing logic and sampling distribution features. These are both advanced stochastic ideas with which students have many difficulties. Moreover randomness is assumed in building a sampling distribution or in hypothesis testing, so a sound understanding of these two concepts in turn actually rests on the idea of randomness (thus creating a circular situation).

Stochastics is a field in which the need to present advanced concepts to a wide audience of students with varied backgrounds, interests, and capacities is urgent, given the implications of poor stochastic thinking in many fields of human activity. For example, recommendations to substitute or complement statistical tests with confidence intervals (e.g., Wilkinson, 1999) do not take into account the fact that their appealing feature is based on a fundamental misunderstanding (Lecoutre, 1998), which consists of thinking of the parameters as random variables and assuming that confidence intervals contain the parameters with a specified probability. Such interpretation is incorrect in a classical inferential framework, although it is acceptable in Bayesian statistics. For this reason some researchers (e.g., Lecoutre & Lecoutre, 2001) are suggesting to change the practice of statistics towards Bayesian methods and suggest that Bayes's thinking is more intuitive than frequentist probability for students and better reflects students' everyday thinking about uncertainty. Reported results from research focused on teaching Bayesian statistics are limited to a few cases (Albert, 2000), and the decision to change from classical to Bayesian approach is also dependent on researchers' own objective or subjective views of probability. The wide research done on students' and professionals' misunderstanding and misuse of advanced statistics should be complemented by a similar effort in designing and evaluating teaching experiments oriented to help students and researchers overcome these difficulties. In this sense, statistics and probability can be a paradigm for finding ways and approaches to introduce advanced mathematical ideas to wide audiences and to rethink the very meaning of what is advanced mathematical thinking.

CONCLUSION

In this chapter, we have tried to synthesize the evolution of research on mathematics thinking and learning at post-secondary level since the first *Handbook* was published in 1992, and to analyze its most important advances, their potential, and limit for understanding and improving teaching and learning processes at this advanced level. This evolution, which has been fostered by both internal and external factors, has not obeyed a simple dynamic, and the multiplicity of its facets reflects both the intrinsic diversity of educational research and the diversity of the changes that have affected post-secondary education during the last decade. Some of these evolutions were highly predictable, such as the development of the several theories of reification focusing on process-object duality; the increasing attention paid to the semiotic dimension of mathematical activity and to the essential role played by connections between representations, settings, and perspectives in mathematical thinking and learning; and the increasing influence taken by sociocultural and anthropological approaches towards learning processes. Others were less predictable, such as the increasing theoretical influence of current developments in cognitive sciences and embodied cognition, the rapid changes and growth in mathematical practice fostered by information and communication technologies, the reflection of these changes in post-secondary education mathematical curricula that today oblige educational researchers to expand their privileged fields of investigation, the increasing demand of advanced mathematics learning by varied types of students, and the interest in research on teaching and learning mathematics from a variety of disciplines (not just in mathematics education). All these evolutions make the landscape of research on mathematics thinking and learning at post-secondary level today something much more diverse and richer than was the case about 10 years ago.

Looking at what has been achieved, we have the feeling that the research initiated in the Advanced Mathematical Thinking working group of PME in this area, in spite of the diversity of its developments, has avoided a fractionalization of its perspectives and been able to integrate its previous achievements into complementary and coherent constructs, as we have tried to show in the first parts of this chapter. This certainly provides researchers with a strong and mature basis for addressing the important challenges that research has to face today.

More and more, however, the evolution of post-secondary education obliges us to reconsider the answers we have given to fundamental epistemological issues about the nature of mathematics, and the nature of mathematical learning and thinking. These fundamental issues cannot be discussed without taking into account the current reality of mathematical practices, or by considering only the practice of pure mathematicians working in traditional fields. Post-secondary educational research has from this point of view a specific epistemological role to play in educational research thanks to its proximity with the professional world of mathematics. The increasing importance taken in post-secondary mathematics education by service courses faces us with the necessity of taking a wider perspective. As has been shown in the fourth part of this chapter, we are pushed and questioned by the evolution of mathematical practices in professional fields such as engineering. In the case of statistics, we observe a progressive separation of mathematics from the applications of statistics that originates in the fact that statistics has changed much more rapidly than mathematics, is much more dependent on the context and on information technology, and is usually taught at post-secondary level by lecturers who are seldom mathematicians. At the same time, an increasing amount of research is being carried out in advanced stochastics outside the mathematics education community.

Technology has deeply changed professional practices and mathematical needs and as a consequence changed what has to be learnt and how it can be learnt. As has been argued for the case of stochastics and engineering, these changes oblige us to see in technology more than an educational help, but something constitutive of mathematical practices, and moreover having the effect of changing mathematical practices as well as changing the meaning of mathematical objects. This has an impact on the learning processes and obliges us also to consider forms of and progressions in learning that are different from the usual ones. At the same time these changed practices should support ways for students with a limited mathematical background to make reasonable sense and use of the very sophisticated mathematics that are implemented in the technology they use, thus producing new, more intuitive meanings for these mathematical objects.

From the theoretical point of view, researchers in these new research fields have taken into account only a very limited number of the theoretical concepts and paradigms developed by the post-secondary mathematics education community. Therefore the study of the extent to which these frameworks can describe the learning of service mathematics or the learning of advanced stochastics or how these frameworks should be complemented with other constructs specific to these fields is still an open challenge for researchers.

Thanks to the global evolution of theoretical frames towards sociocultural and anthropological perspectives, researchers are certainly today better equipped for addressing this issue of mathematical practices, taking into account their different components, both explicit and tacit, in order to analyze their potential cognitive effects. This evolution has also a corollary that any consideration of learning processes is necessarily relative. Knowledge emerges from practices, and the learning processes one can access are those that are made possible by the existing practices. We have thus to be aware that the answers we can provide to the questions at stake are not absolute; they depend on the current state of educational practice. Research has certainly to pay more attention to these dependences than it has done in the past. This makes it necessary to explore more systematically than has been done before what is learnt and how in educational designs different from the traditional ones, as for instance those mentioned in the fourth part of the chapter where the learning of mathematics emerges from the realization of projects and activities in which modeling and technology are given an essential role.

Even if researchers seem today better equipped for approaching the relationships between learning and practices, and sensitive to the cognitive diversity that emerges from their diversity, benefiting today from the different advances is not at all an easy task. We met this difficulty when writing this chapter. For instance, we found it essential to open the chapter to challenging areas such as stochastics or engineering, and our initial project was to end this chapter by some kind of integrated perspective on learning and thinking at post-secondary level. But connecting research in stochastics and research on AMT, two areas of research that have developed in isolated ways and pushed by different logics, turned out to be too difficult. Much more work than what we were able to do within the time devoted to the elaboration of this chapter would have been necessary in order to succeed. What is certainly expressed in this chapter is the specific coherence underlying each of these approaches, and the ways it tends to question the other one, but no more.

We end this chapter with the feeling that research on mathematical learning and thinking at post-secondary level is entering now a new and fascinating phase, with difficult and new challenges to face, challenges that will require to be solved to extend interactions and collaborations beyond the traditional community of research in mathematics education at advanced level.

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AUTHOR NOTE

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