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THE ROLE OF VISUAL REPRESENTATIONS IN THE LEARNING OF MATHEMATICS ⋆

ABSTRACT. Visualization, as both the product and the process of creation, interpretation and reflection upon pictures and images, is gaining increased visibility in mathematics and mathematics education. This paper is an attempt to define visualization and to analyze, exemplify and reflect upon the many different and rich roles it can and should play in the learning and the doing of mathematics. At the same time, the limitations and possible sources of difficulties visualization may pose for students and teachers are considered.

INTRODUCTION

Vision is central to our biological and socio-cultural being. Thus, the biological aspect is described well in the following (Adams and Victor, 1993, p. 207): "The faculty of vision is our most important source of information about the world. The largest part of the cerebrum is involved in vision and in the visual control of movement, the perception and the elaboration of words, and the form and color of objects. The optic nerve contains over 1 million fibers, compared to 50,000 in the auditory nerve. The study of the visual system has greatly advanced our knowledge of the nervous system. Indeed, we know more about vision than about any other sensory system". As for the socio-cultural aspect, it is almost a commonplace to state that we live in a world where information is transmitted mostly in visual wrappings, and technologies support and encourage communication which is essentially visual. Although "people have been using images for the recording and communication of information since the cave-painting era ... the potential for 'visual culture' to displace 'print culture' is an idea with implications as profound as the shift from oral culture to print culture." (Kirrane, 1992, p. 58).

Therefore, as biological and as socio-cultural beings, we are encouraged and aspire to 'see' not only what comes 'within sight', but also what

⋆ Plenary lecture at the North American Chapter of the International Group for the Psychology of Mathematics Education Conference (PME-NA), Cuernavaca, México, 1999, appeared in Hitt, F. and Santos, M. (eds.), *Proceedings of the 21st Annual Meeting of PME-NA*, ERIC, Clearinghouse for Sciece, Mathematics and Environmental Education, OH, pp. 55–80.

Educational Studies in Mathematics **52:** 215–241, 2003.

we are unable to see. Thus, one way of characterizing visualization and its importance, both as a 'noun' – the product, the visual image – and as a 'verb' – the process, the activity – (Bishop, 1989, p. 7), is that "Visualization offers a method of seeing the unseen" (McCormick et al., 1987, p. 3). I take this sentence as the leitmotif of this presentation in order to re-examine first, its nature and its role and then, the innovations of research and curriculum development.

A short digression to the point: What about blind mathematicians?[1] Dealing at length with this question is beyond the scope of this paper. However, it is interesting to note that, as implied above and as wonderfully described in Jackson (2002), visualization may go far beyond the unimpaired (physiological) sense of vision.

SEEING THE UNSEEN – A FIRST ROUND

Taken literally, the unseen refers to what we are unable to see because of the limitations of our visual hardware, e.g. because the object is too far or too small. We have developed technologies to overcome these limitations to make the unseen seeable. Consider, for example, the photographs taken by Pathfinder on Mars in 1997. Or, for example, a 4,000 times amplification of a white blood cell about to phagocytise a bacterium, or a 4,000 times amplification of a group of red blood cells. We may have heard descriptions of them prior to seeing the pictures, and our imagination may have created images for us to attach to those descriptions. But seeing the thing itself, with the aid of technology which overcomes the limitation of our sight, provides not only a fulfillment of our desire to 'see' and the subsequent enjoyment, but it may also sharpen our understanding, or serve as a springboard for questions which we were not able to formulate before.

SEEING THE UNSEEN – IN DATA

In a more figurative and deeper sense, seeing the unseen refers to a more 'abstract' world, which no optical or electronic technology can 'visualize' for us. Probably, we are in need of a 'cognitive technology' (in the sense of Pea, 1987, p. 91) as "any medium that helps transcend the limitations of the mind ... in thinking, learning, and problem solving activities." Such 'technologies' might develop visual means to better 'see' mathematical concepts and ideas. Mathematics, as a human and cultural creation dealing with objects and entities quite different from physical phenomena (like planets or blood cells), relies heavily (possibly much more than mathematicians would be willing to admit) on visualization in its different forms and at different levels, far beyond the obviously visual field of geometry, and spatial visualization. In this presentation, I make an attempt to scan through these different forms, uses and roles of visualization in mathematics education. For this purpose, I first blend (and paraphrase) the definitions of Zimmermann and Cunningham (1991, p. 3) and Hershkowitz et al. (1989, p. 75) to propose that:

> Visualization is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings.

A first type of the 'unseen' we find in mathematics (or allied disciplines, e.g. data handling or statistics) consists of data representations.

The following example is a chart, considered a classic of data graphing, designed by Charles Joseph Minard (1781–1870), a French engineer.

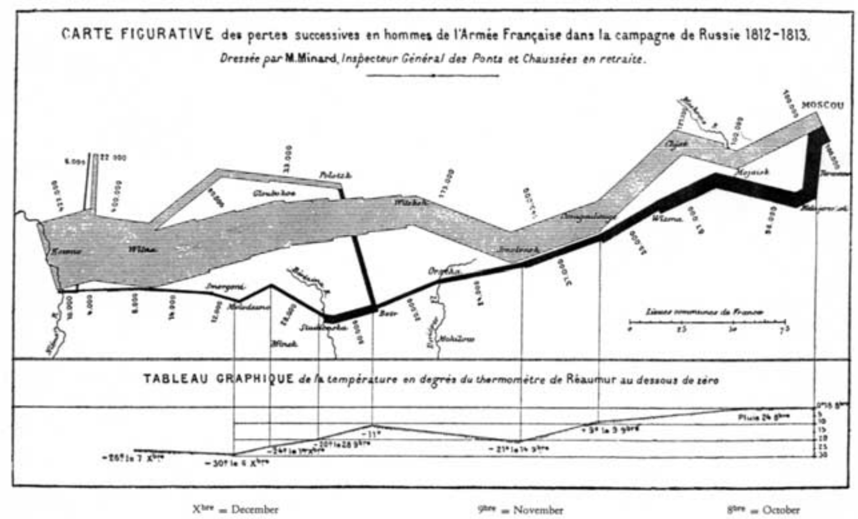

Figure 1. A chart describing Napoleon's 1812 Russian campaign.

Tufte (1983, p. 40) considers this graph a "Narrative Graphic of Space and Time", and refers to Marey (1885, p. 73) who notes that it defies the historian's pen by its brutal eloquence in portraying the devastating losses suffered in Napoleon's 1812 Russian campaign. At the left, at the then Polish-Russian border, the chart shows the beginning of Napoleon's campaign with an army of 422,000 men (represented by the width of the 'arm'), the campaign itself and the retreat (black 'arm'), which is connected to a sub-chart showing dates and temperatures. The two-dimensional graph tells the ‘whole’ story by displaying *six* variables: the army size, its exact (two-dimensional) location, direction, temperature and dates in a compact and global condensation of information. The visual display of information enables us to ‘see’ the story, to envision some cause-effect relationships, and possibly to remember it vividly. This chart certainly is an illustration of the phrase “a diagram is worth a thousand (or ‘ten thousand’) words,” because of a) their two-dimensional and non-linear organization as opposed to the emphasis of the ‘printed word’ on sequentiality and logical exposition (Larkin and Simon, 1987, p. 68, Kirrane, 1992, p. 59); and b) their grouping together of clusters of information which can be apprehended at once, similarly to how we see in our daily lives, which helps in “reducing knowledge search” (Koedinger, 1992, p. 6) making the data “perceptually easy” (Larkin and Simon, 1987, p. 98).

Anscombe (1973, p. 17) claims that “Graphs can have various purposes, such as: (i) to let us perceive and appreciate some broad features of the data, (ii) to let us look behind those broad features and see what else is there”. And he presents an example of four sets of data (Figure 2) that can be characterized by a set of identical parameters (Figure 3) and which looks quite different when the data are plotted (Figure 4):

I		II		III		IV	
X	Y	X	Y	X	Y	X	Y
10.0	8.04	10.0	9.14	10.0	7.46	8.0	6.58
8.0	6.95	8.0	8.14	8.0	6.77	8.0	5.76
13.0	7.58	13.0	8.74	13.0	12.74	8.0	7.71
9.0	8.81	9.0	8.77	9.0	7.11	8.0	8.84
11.0	8.33	11.0	9.26	11.0	7.81	8.0	8.47
14.0	9.96	14.0	8.10	14.0	8.84	8.0	7.04
6.0	7.24	6.0	6.13	6.0	6.08	8.0	5.25
4.0	4.26	4.0	3.10	4.0	5.39	19.0	12.50
12.0	10.84	12.0	9.13	12.0	8.15	8.0	5.56
7.0	4.82	7.0	7.26	7.0	6.42	8.0	7.91
5.0	5.68	5.0	4.74	5.0	5.73	8.0	6.89

Figure 2. Four sets of data.

In this case, the graphical display may support the unfolding of dormant characteristics of the data (hidden by the identical and somehow not trans-

N = 11
mean of X's = 9.0
mean of Y's = 7.5
equation of regression line: $Y = 3 + 0.5X$
standard error of estimate of slope = 0.118
t = 4.24
sum of squares $X - \overline{X} = 110.0$
regression sum of squares = 27.50
residual sum of squares of Y = 13.75
correlation coefficient = .82
$r^2 = .67$

Figure 3. The data parameters.

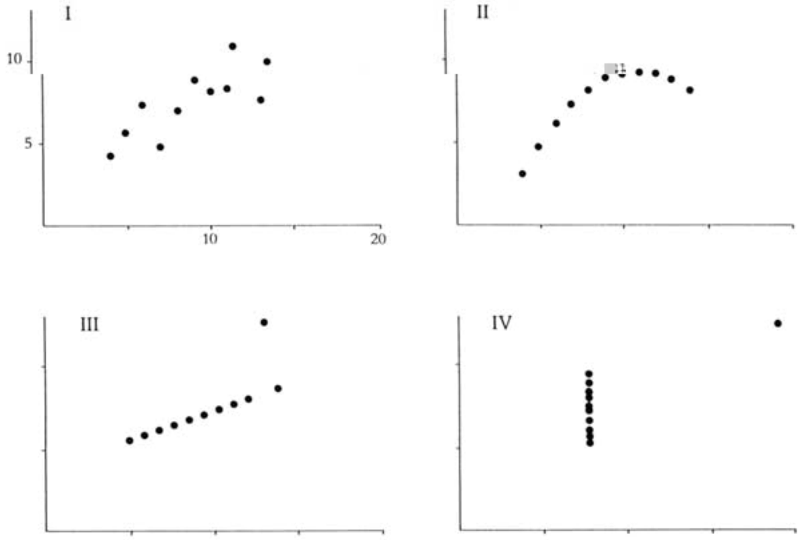

Figure 4. The data plots.

parent parameters), because it does more than just depict. As Tufte says, that in this case: "Graphics *reveal* data. Indeed graphics can be more precise and revealing than conventional statistical computations".

SEEING THE UNSEEN – IN SYMBOLS AND WORDS

Visualization can accompany a symbolic development, since a visual image, by virtue of its concreteness, can be "an essential factor for creating the feeling of self-evidence and immediacy" (Fischbein, 1987, p. 101).

Consider, for example, the mediant property of positive fractions: $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$. Flegg, Hay and Moss (1985, p. 90) attribute this 'rule of intermediate numbers' to the French mathematician Nicolas Chuquet, as it appears in his manuscript La Triparty en la Science des Nombres (1484). The symbolic proof of this property is quite simple, yet it may not be very illuminating to students. Georg Pick (1859–1943?) an Austrian-Czech mathematician wrote "The plane lattice ... has, since the time of Gauss, been used often for visualization and heuristic purposes. ... [in this paper] an attempt is made to put the elements of number theory, from the very beginning, on a geometrical basis" (free translation from the original German in Pick, 1899). Following Pick, we represent the fraction $\frac{a}{b}$ by the lattice point *(b,a)*. The reason for representing $\frac{a}{b}$ by *(b,a)* and not *(a,b)*, is for visual convenience, since the slope of the line from the origin O to *(b,a)* is precisely $\frac{a}{b}$, and hence fractions arranged in ascending order of magnitude are represented by lines in ascending order of slope. Visually, the steeper the line the larger the fraction. (Note also that equivalent fractions are represented by points on the same line through the origin. If the lattice point P represents a reduced fraction, then there are no lattice points between O and P on the line OP). Now, the visual version of the mediant of $\frac{1}{3}$, $\frac{4}{5}$, which is $\frac{1+4}{3+5}$, is represented by the diagonal of the parallelogram 'defined' by $\frac{1}{3}$ and $\frac{4}{5}$.

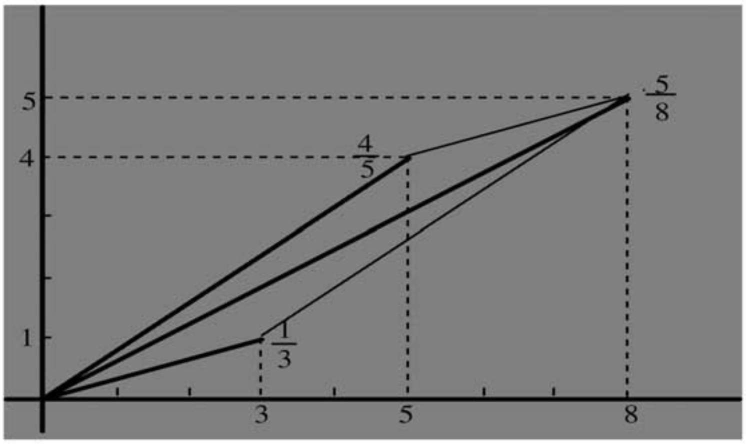

Figure 5. Visual representation of a mediant.

The same holds in general for $\frac{a}{b}$, $\frac{c}{d}$. I would claim that the 'parallelogram' highlights the simple idea of betweeness and also the reason for the property, and thus it may add meaning and conviction to the symbolic proof. In this example, we have not only represented the fraction $\frac{a}{b}$ visually by the point with coordinates *(b,a)* – or the line from the origin through *(b,a)* – but capitalized on the visualization to bring geometry to the aid of what seem to be purely symbolic/algebraic properties. Much mathematics can be done on this basis; see, for example Bruckheimer and Arcavi (1995).

In a similar spirit, Papert (1980, p. 144) brings the following problem. "Imagine a string around the circumference of the earth, which for this purpose we shall consider to be a perfectly smooth sphere, four thousand miles in radius (6,400 km approximately). Someone makes a proposal to place a string on six-foot-high (about 1.8 meters) poles. Obviously this implies that the string will have to be longer." How much longer? Papert says that "Most people who have the discipline to think before calculating ... experience a compelling intuitive sense that 'a lot' of extra string is needed." However, the straightforward algebraic representation yields $2\pi(R+h)–2\pi R$, where R is the radius of the Earth and h the height of the poles. Thus the result is $2\pi h$, less than 38 feet (less than 12 meters), which is a) amazingly little and b) independent of the radius of the Earth!

For many, this result is a big surprise, and a cause for reflection on the gap between what was expected and what was obtained. Papert was uncomfortable with the possible morale from this example, that our initial intuitions may be faulty, therefore they should not be trusted, and it is only the symbolic argument that should count. His discomfort led him to propose a visual solution, which would serve to educate, or in his own words to 'debug', our intuitions, so that the symbolic solution is not only regarded as correct, but also natural and intuitively convincing. His non-formal and graphical solution starts with a simple case, a string around a 'square Earth'.

"The string on poles is assumed to be at distance h from the square. Along the edges the string is straight. As it goes around the corner it follows a circle of radius h... The extra length is all at the corners... the four quarter circles make a whole circle... that is to say $2\pi h$." (p. 147). If we increase the sides of the square, the amount of extra string needed is still the extra four quarters of a circle with radius $2\pi h$. Then he proceeds to deform 'continuously' the square towards the round earth. First by looking at the shape of an octagon.

The extra pieces of string "is all in the pie slices at the corners. If you put them together they form a circle of radius h. As in the case of the

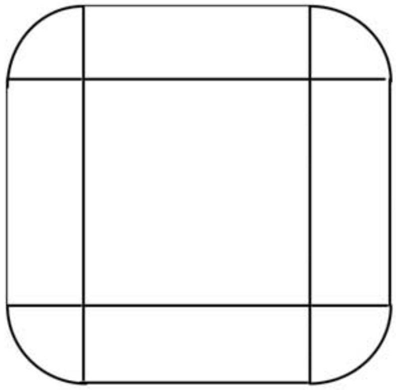

Figure 6. A string around a 'square Earth'.

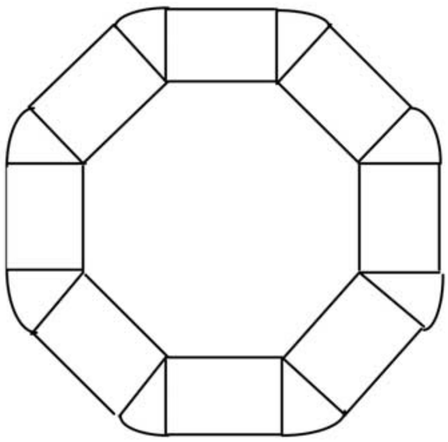

Figure 7. A string around an 'octagonal Earth'.

square, this circle is the same whether the octagon is small or big. What works for the square (4-*gon*) and for the octagon (8-*gon*) works for the 100-*gon* and for the 1000-*gon*." (p. 149). The formal symbolic result becomes now also visually (and thus intuitively) convincing. After such a solution, we may overhear ourselves saying "I see", double-entendre intended. Visualization here (and in many similar instances) serves to adjust our 'wrong' intuitions and harmonize them with the opaque and 'icy' correctness of the symbolic argument.

Another role of visualization in an otherwise 'symbolic' context, is where the visual solution to a problem may enable us to engage with concepts and meanings which can be easily bypassed by the symbolic solution of the problem. Consider, for example, the following:

What is the common characteristic of the family of linear functions whose equation is $f(x)=bx+b$?

The symbolic solution would imply a simple syntactic transformation and its interpretation: $f(x)=bx+b= b(x+1)$ – regardless of the value of b, for all these functions, f(–1)=0 – or in other words, all share the pair $(-1,0)$. Compare this to the following graphical solution, produced by a student: In $f(x)=bx+b$, the first b is the slope, the second is the y-intercept. Since slope is 'rise over run', and since the value of the slope is the same value as the y-intercept, to a rise with the value of the y-intercept must correspond a run of *1* [2].

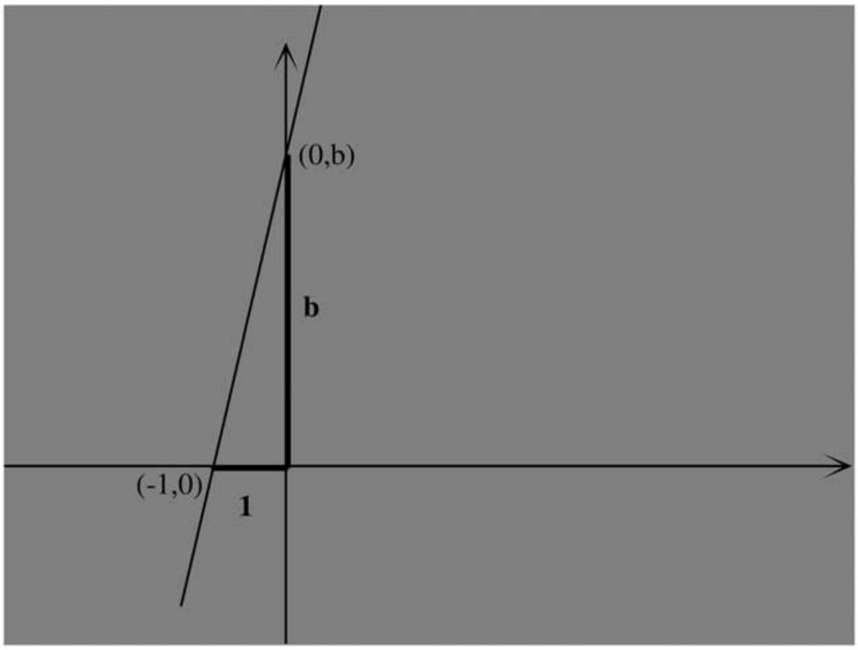

Figure 8. A linear function of the form f(x)=bx+b.

Sophisticated mathematicians may claim to 'see' through symbolic forms, regardless of their complexity. For others, and certainly for mathematics students, visualization can have a powerful complementary role in the three aspects highlighted above: visualization as (a) support and illustration of essentially symbolic results (and possibly providing a proof in its own right) as in the case of the mediant property, (b) a possible way of resolving conflict between (correct) symbolic solutions and (incorrect) intuitions, as with the string around the earth problem, and (c) as a way to help us re-engage with and recover conceptual underpinnings which may be easily bypassed by formal solutions, as with the slope = intercept task.

FORESEEING THE UNSEEN – AT THE SERVICE OF PROBLEM SOLVING

Davis (1984, p. 35) describes a phenomenon which he calls *visually-moderated sequences* (VMS). VMS frequently occurs in our daily lives. Think of the "experience of trying to drive to a remote location visited once or twice years earlier. Typically, one could not, at the outset, tell anyone how to get there. What one hopes for is, . . ., a VMS. . .: see some key landmark . . . and hope that one will remember what to do at the point. Then one drives on, again hoping for a visual reminder that will cue the retrieval of the next string of remembered directions." In this case, visualization may function as a tool to extricate oneself from situations in which one may be uncertain about how to proceed. As such it is linked, in this case, not so much to concepts and ideas, but rather to perceptions which lead procedural decisions. One of the mathematical examples Davis (p. 34) brings is the following: "A student asked to factor $x^2 - 20x + 96$, might ponder for a moment, then write

$x^2 - 20x + 96$
(   ) (   ),
then ponder, then write,
$x^2 - 20x + 96$
(x   ) (x   ),
then ponder some more, then continue writing
$x^2 - 20x + 96$
(x –   ) (x –   ),
and finally complete the task as
$x^2 - 20x + 96$
$(x - 12)\ (x - 8)$"

The mechanism is more or less: "look, ponder, write, look, ponder, write, and so on." In other words, "a visual clue V_1 elicits a procedure P_1 whose execution produces a new visual cue V_2, which elicits a procedure P_2,. . . and so on."

Visualization at the service of problem solving, may also play a central role to inspire a whole solution, beyond the merely procedural. Consider, for example, the following problem (Barbeau, 1997, p. 18).

Let n be a positive integer and let an nxn square of numbers be formed for which the element in the ith row and the jth column ($1 \leqslant i, j \leqslant n$) is the smaller of i and j. For $n = 5$, the array would be:

1	1	1	1	1
1	2	2	2	2
1	2	3	3	3
1	2	3	4	4
1	2	3	4	5

Figure 9. A number array.

Show that the sum of all the numbers in the array is $1^2 + 2^2 + 3^2 + \ldots n^2$.

A solution.

1	1	1	1	1
1	2	2	2	2
1	2	3	3	3
1	2	3	4	4
1	2	3	4	5

Figure 10. An approach to finding the sum of the number array.

"Algebraically, we see that the sum of the numbers in the kth gnomon consisting of the numbers not exceeding k in the kth row and the kth column is $(1 + 2 + \ldots + (k–1))\ 2 + k = k^2$. The result follows." (Barbeau, p. 20)

This solution has some elements of visualization in it: it identifies the gnomons as 'substructures' of the whole in which a clear pattern can be established. However, an alternative solution presented by the author is even more interesting visually.

"We can visualize the result by imagining an nxn checkerboard. Begin by placing a checker on each square (n^2 checkers); place an additional checker on every square not in the first row or the first column ($(n–1)^2$ checkers); then place another checker on every square not in the first two rows or the first two columns ($(n–2)^2$ checkers). Continue on in this way to obtain an allocation of $n^2 + (n–1)^2 + (n–2)^2 + \cdots + 2^2 + 1^2$ checkers; the number of checkers placed on the square in the ith row and the jth column is the smaller of i and j." (See also Arcavi and Flores, 2000).

In the examples in previous subsections, visualization consisted of making use of a visual representation of the problem statement. I claim that,

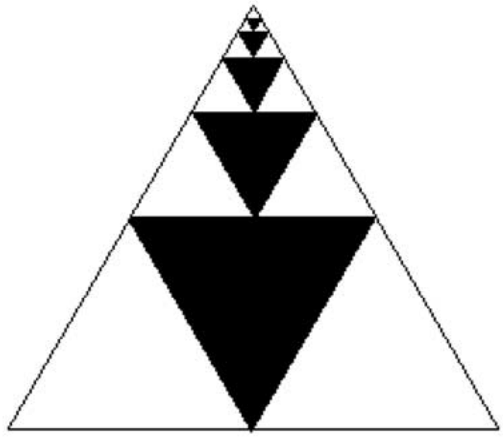

Figure 11. A visual representation for the sum of the series.

in this example, visualization consists of more than just a translation, the solver imagined a strongly visual 'story' (not at all implied by the problem statement), he imposed it on the problem, and derived from it the solution.

Probably the inspiration for this visual story, was the author's previous experience and knowledge, which helped him envision the numerical values in a matrix as height, or, in other words, he probably saw a 2-D compression of a 3-D data representation. In any case, one's visual repertoire can fruitfully be put at the service of problem solving and inspire creative solutions.

SEEN THE UNSEEN – MORE THAN JUST BELIEVING IT? PERHAPS ALSO PROVING IT?

"Mathematicians have been aware of the value of diagrams and other visual tools both for teaching and as heuristics for mathematical discovery. ... But despite the obvious importance of visual images in human cognitive activities, visual representation remains a second-class citizen in both the theory and practice of mathematics. In particular, we are all taught to look askance at proofs that make crucial use of diagrams, graphs, or other non-linguistic forms of representation, and we pass on this disdain to students." However, "visual forms of representation can be important ... as legitimate elements of mathematical proofs." (Barwise and Etchemendy, 1991, p. 9). We have already illustrated this in the example of the mediant property of fractions. As another example, consider the following beautiful proof taken from the section bearing the very suggestive name of "proofs without words" (Mabry, 1999, p. 63) $\frac{1}{4} + (\frac{1}{4})^2 + (\frac{1}{4})^3 + \cdots = \frac{1}{3}$ (Figure 11)

It can be argued that the above is neither (a) 'without words' nor (b) 'a proof'. Because (a) although verbal inferences are not explicit, when we see it, we are most likely to decode the picture by means of words (either aloud or mentally); and (b) Hilbert's standard for a proof to be considered as such is whether it is arithmetizable, otherwise it would be considered non-existent (Hadamard, 1954, p. 103). As to the first reservation, we may counter-argue that visualization as a process is not intended to exclude verbalization (or symbols, or anything else), quite the contrary, it may well complement it. As to the second reservation, there is a "clearly identifiable if still unconventional movement ... growing in the mathematics community, whose aim is to make visual reasoning an acceptable practice of mathematics, alongside and in combination with algebraic reasoning. According to this movement, visual reasoning is not meant only to support the discovery of new results and of ways of proving them, but should be developed into a fully acceptable and accepted manner of reasoning, including proving mathematical theorems" (Dreyfus, 1994, p. 114).

Consider, for example, the following.

How many matches are needed to build the following nxn square?

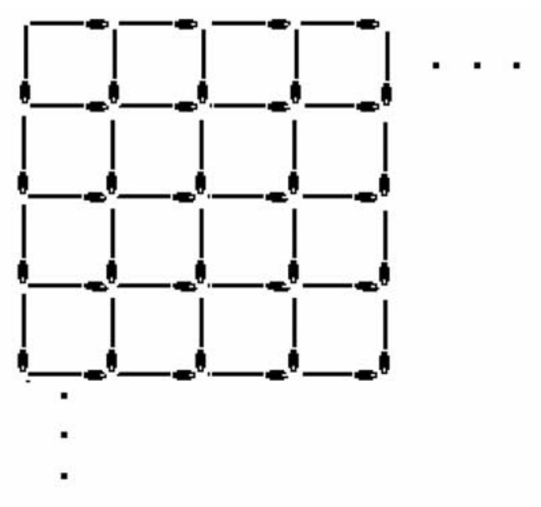

Figure 12. An array of matches.

This problem was tried in several teacher courses in various countries and with several colleagues, and the many solutions proposed were collected and analyzed (Hershkowitz, Arcavi and Bruckheimer, 2001). The majority of the solution approaches were visual, yet they differ in their nature. There were those who decomposed the whole array of matches into what they saw as easily countable units. For example, a square, U's and L's,

Figure 13. Some countable units.

in order to decompose the whole as follows.

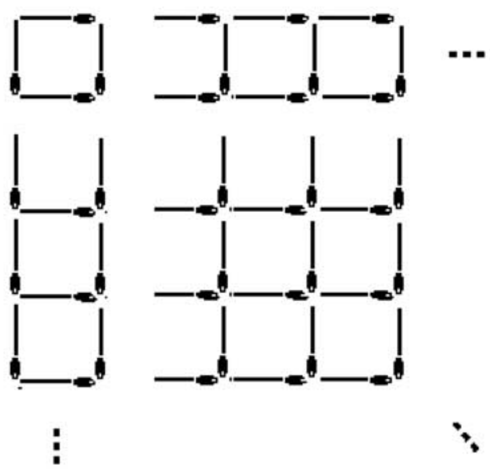

Figure 14. A first possible decomposition.

Others identified L's and single matches, and proposed another decomposition of the whole (Figure 15).

Some participants counted unit squares, and then proceeded to adjust for what was counted twice. Yet others identified the smallest possible unit, a single match and counted the *n* matches in a row (or a column), multiplied it by the *n+1* rows (or columns), and then multiplied by *2*. It would seem that the simpler the visually identified unit (one match), the more global and uniform the counting strategy became. Some participants imagined units whose existence is only suggested: the 'intersection' points (Figure 16) and then proceeded to make auxiliary constructions to make the count uniform (Figure 17) which then they adjusted for double counting and for the added auxiliary matches.

Thus decomposition into what was perceived as easily countable units took different forms, but it was not the only visual strategy. Another visual

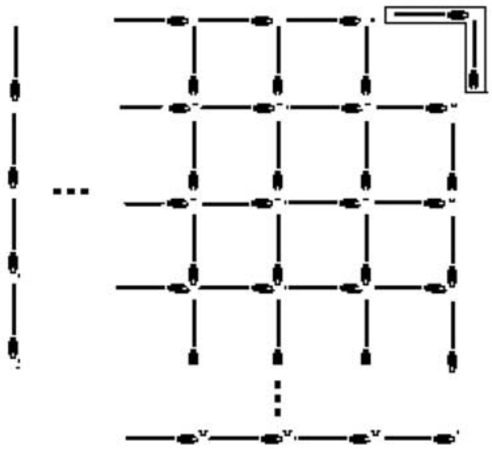

Figure 15. A second possible decomposition.

Figure 16. An 'intersection'.

strategy consisted of changing the whole gestalt into a new one, in which patterns seemed easier for the solver to identify (Figure 18).

Change of gestalt took other forms as well: instead of 'breaking and rearranging' the original whole as above, some imposed an 'auxiliary construction' (Figure 19) whose role consists of providing visual 'crutches', which in themselves are not counted, but which support and facilitate the visualization of a pattern that suggests a counting strategy, yielding $(2+4+\ldots+2n) \times 2$.

Surprisingly, visualization, for others, was sparked by the symbolic result obtained by themselves in a previous attempt or by others. Having obtained the final count in the form $2n(n+1)$, they applied a symbolic transformation to obtain $4 \times \frac{n(n+1)}{2}$ (4 times the sum of 1+2+3+…+n, which they knew to be $\frac{n(n+1)}{2}$). This symbolic transformation inspired the search for a visual pattern for which this formula is the expression of the counting strategy. This exemplifies how visual reasoning can also be inspired, guided and supported, by a symbolic expression, with a good dose of "symbol sense" (Arcavi, 1994).

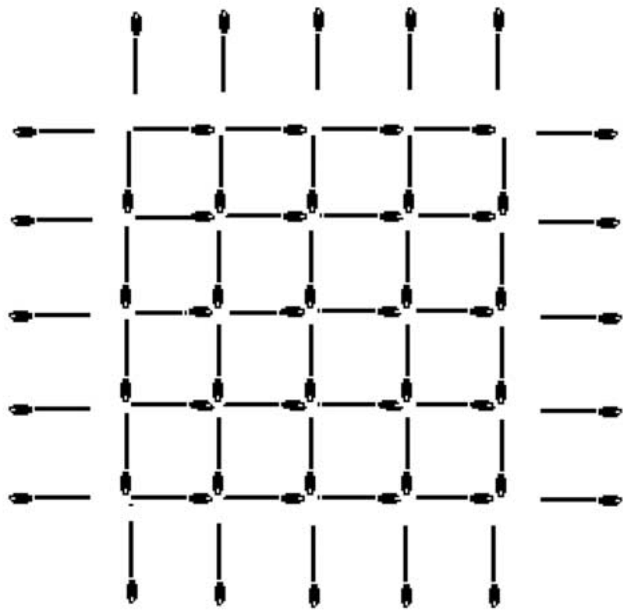

Figure 17. All ‘intersections’.

In sum, we found that, in this task, visualization consisted of processes different in nature. However, all of them seem to corroborate Fischbein’s claim that visualization “not only organizes data at hand in meaningful structures, but it is also an important factor guiding the analytical development of a solution.” (Fischbein, 1987, p. 101). We propose that visualization can be even more than that: it can be the analytical process itself which concludes with a solution which is general and formal.

DO YOU AND I SEE ALIKE?

“We don’t know what we see, we see what we know”. I was told that this sentence is attributed to Goethe. Its last part: “We see what we know” applies to many situations in which students do not necessarily see what we as teachers or researchers do. For some this sentence may be a truism, already described in many research studies, nevertheless it is worth analyzing some examples.

Consider the following taken from Magidson (1989). While working with a graphing software, students were required to type in equations one at a time, and draw their graphs. The equations were $y = 2x+1$, $y = 3x+1$, $y = 4x+1$, and students were asked what do they notice, in which way

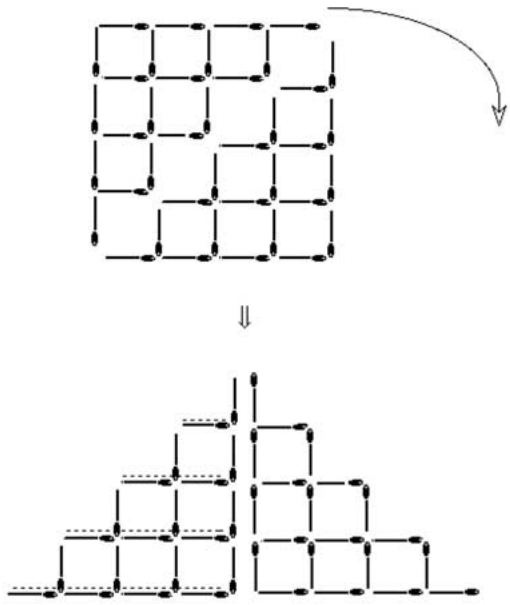

Figure 18. Changing the gestalt by breaking and rearranging.

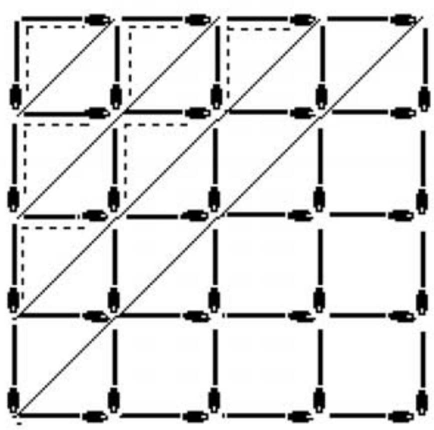

Figure 19. An 'auxiliary construction'.

the lines graphed are similar and in which way they are different, and to predict (and test the prediction) for the graph of $y = 5x+1$. The expectation was that the task would direct student attention (at least at the phenomenological level) to what an expert considers relevant: the influence of the number that multiplies the x, and that all lines go through *(0,1)*. Presmeg (1986, p. 44) stated that many times "an image or diagram may tie thought to irrelevant details", irrelevant to an expert, that is. Some of the answers reported by Magidson certainly confirm this: there were students who 'noticed' the way the software draws the lines as 'starting' from the bottom of the screen. Others talked about the degree of jaggedness of the lines, which is an artifact of the software and depends on how slanted the lines are. And there were those who noticed that the larger the number, the more 'upright' the line, but when asked to predict the graph of $y = 5x+1$, their sketch clearly did not go through *(0,1)*.

A similar phenomenon in a slightly different context is reported by Bell and Janvier (1981) where they describe 'pictorial distractions': graphs are judged by visually salient clues, regardless of the underlying meanings.

Clearly, our perception is shaped by what we know, especially when we are looking at what Fischbein (as reported in Dreyfus, 1994, p. 108) refers to as diagrams which are loaded with an 'intervening conceptual structure'. Some of the visual displays I have brought so far are either displays of objects (matches) or arrays of numbers which allow one to observe and manipulate patterns. Others were displays of data, for which a small number of ad-hoc conventions suffices to make sense of the graph. However, when we deal, for example, with Cartesian graphs of linear functions, what we look at has an underlying representation system of conceptual structures. Experts may often be surprised that students who are unfamiliar (or partially familiar) with the underlying concepts see 'irrelevancies' which are automatically dismissed (or even unnoticed) by the expert's vision.

I would like to claim further: in situations like the one described by Magidson, what we see is not only determined by the amount of previous knowledge which directs our eyes, but in many cases it is also determined by the context within which the observation is made. In different contexts, the 'same' visual objects may have different meanings even for experts. Consider for example, the diagram in Figure 20.

What we see are three parallel lines. If nothing more is said about the context, we would probably think about the Euclidean geometry associations of parallelism (equal distance, no intersection, etc.). Consider now the same parallel lines, with a superimposed Cartesian coordinate system.

For a novice, this may be no more than the same picture with two extra lines. For experts, this would probably trigger much more: the conceptual world of Cartesian representation of functions. The lines are now not only geometrical objects, they have become representations of linear functions, and hence suggest, for example, that to each line corresponds an equation of the form $y = ax+b$ (or any other equivalent form), that these lines have equal slope (share the same a value), and that there is no solution for a pair of equations corresponding to a pair of lines. Moreover, it may also

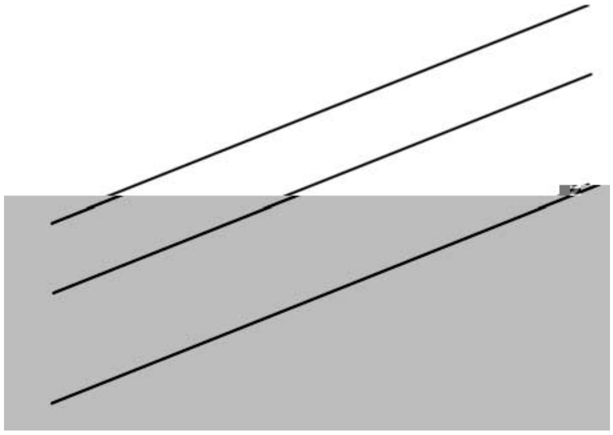

Figure 20. Three parallel lines.

re-direct the attention from the notion of distance between parallel lines (as the length of a segment perpendicular to both) towards the notion of the vertical displacement from one line to the other, which is reflected by the difference in the b values (Figure 21).

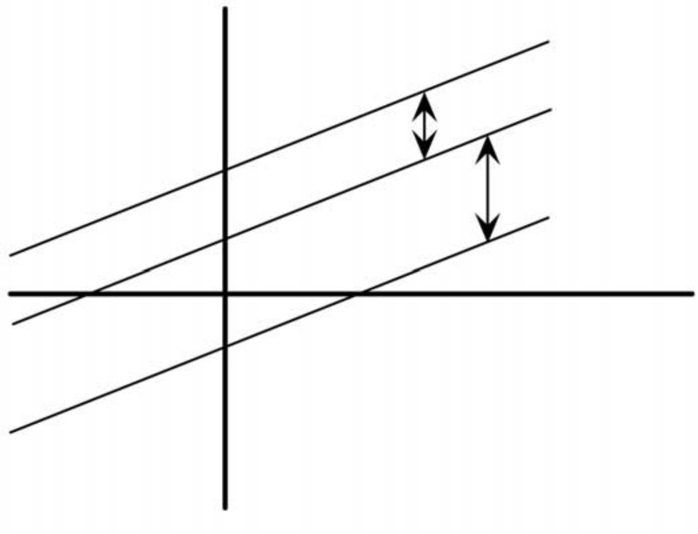

Figure 21. Three parallel lines on the Cartesian plane.

If we now remove the superimposed Cartesian axes and replace them by a system of parallel axes to represent linear functions, experts familiar with such a representation see very different things (Figure 22).

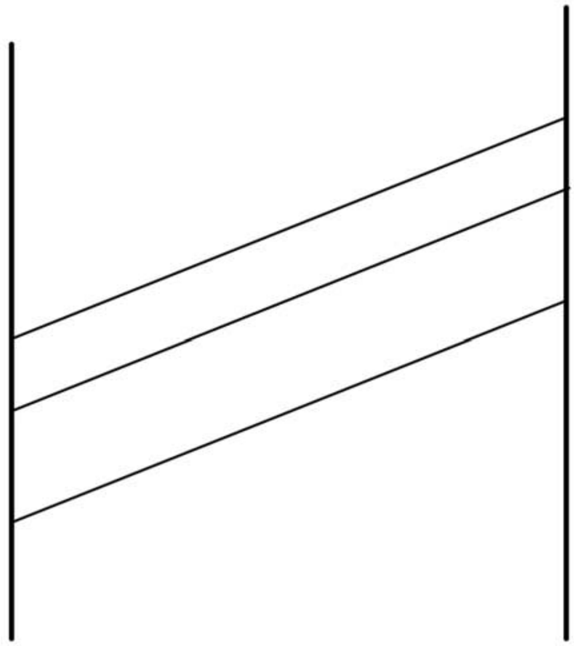

Figure 22. Three parallel segments on the Parallel Axes Representation.

In this case, the three lines are the representation of just three particular ordered pairs of a single linear function (usually the points in the left hand axis representing the domain – equivalent to the x-axis in the Cartesian plane – are put into correspondence with points in the right-hand axis through 'mapping segments'). In this case, the parallelism between the mapping segments indicate that an interval of the domain is mapped onto an interval of equal length in the co-domain, namely that the slope is *1*. (Further details about the Parallel Axes Representation, and its visually salient characteristics, can be found, for example, in Arcavi and Nachmias, 1989, 1990, 1993).

In sum, many times our perceptions are conceptually driven, and seeing the unseen in this case is not just producing/interpreting a 'display that reveals' or a tool with which we can think, as in many examples above. Seen the unseen may refer, as in the examples above, to the development and use of an intervening conceptual structure which enables us to see through the same visual display, things similar to those seen by an expert. Moreover, it also implies the competence to disentangle contexts in which similar objects can mean very different things, even to the same expert.

VISUALIZATION IN MATHEMATICS EDUCATION – SOME UNSEENS WE ARE BEGINNING TO 'SEE'

Nowadays, the centrality of visualization in learning and doing mathematics seems to become widely acknowledged. Visualization is no longer related to the illustrative purposes only, but is also being recognized as a key component of reasoning (deeply engaging with the conceptual and not the merely perceptual), problem solving, and even proving. Yet, there are still many issues concerning visualization in mathematics education which require careful attention.

Borrowing from Eisenberg and Dreyfus (1991), I propose to classify the difficulties around visualization into three main categories: 'cultural', cognitive and sociological.

A 'cultural' difficulty refers to the beliefs and values held about what mathematics and doing mathematics would mean, what is legitimate or acceptable, and what is not. We have briefly referred to this issue while discussing the status of visual proofs. Controversy within the mathematics community, and statements such as "this is not mathematics" (Sfard, 1998, p. 454) by its most prominent representatives, are likely to permeate through to the classroom, via curriculum materials, teacher education etc. and shape their emphasis and spirit. This attitude, which Presmeg (1997, p. 310) calls 'devaluation' of visualization, leaves little room for classroom practices to incorporate and value visualization as a an integral part of doing mathematics.

Cognitive difficulties include, among other things, the discussion whose simplistic version would read as follows: is 'visual' easier or more difficult? When visualization acts upon conceptually rich images (or in Fischbein's words when there are intervening conceptual structures), the cognitive demand is certainly high. Besides, reasoning with concepts in visual settings may imply that there are not always procedurally 'safe' routines to rely on (as may be the case with more formal symbolic approaches). Consciously or unconsciously, such situations may be rejected by students (and possibly teachers as well) on the grounds of being too 'slippery', 'risky' or 'inaccurate'.

Another cognitive difficulty arises from the need to attain flexible and competent translation back and forth between visual and analytic representations of the same situation, which is at the core of understanding much of mathematics. Learning to understand and be competent in the handling of multiple representations can be a long-winded, context dependent, non-linear and even tortuous process for students (e.g. Schoenfeld, Smith and Arcavi, 1993).

Under sociological difficulties, I would include what Eisenberg and Dreyfus (1991) consider as issues of teaching. Their analysis suggests that teaching implies a “didactical transposition” (Chevallard, 1985) which, briefly stated, means the transformation knowledge inexorably undergoes when it is adapted from its scientific/academic character to the knowledge as it is to be taught. It is claimed that this process, by its very nature, linearizes, compartmentalizes and possibly also algorithmetizes knowledge, stripping it (at least in the early stages) from many of its rich interconnections. As such, many teachers may feel that analytic representations, which are sequential in nature, seem to be more pedagogically appropriate and efficient.

Another kind of difficulty under the heading ‘sociological’ (or better socio-cultural), is the tendency of schools in general, and mathematics classrooms in particular, to consist of students from various cultural backgrounds. Some students may come from visually rich cultures, and therefore for them visualization may counteract possible ‘deficits’. In contrast, visualizers may be under-represented amongst high mathematical achievers (Presmeg, 1986, 1989).

Recent curriculum and research studies are taking into account some of the above difficulties and address them, in order to propose and explore innovative approaches to understand and exploit the potential of visually oriented activities. Consider, for example, the following task from Yerushalmy (1993, p. 10),

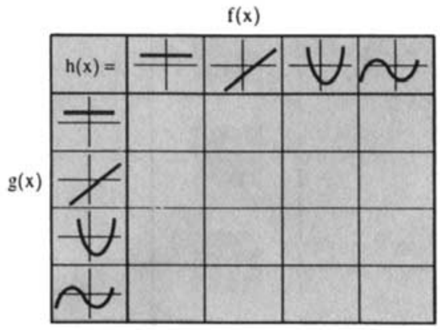

Figure 23. Sketching the quotient of functions.

in which the goal is to sketch the graphs of different types of rational functions of the form h(x) = $\frac{f(x)}{g(x)}$, obtained from the given graphs of f(x) and g(x), and analyze the behavior of the asymptotes (if any).

Arcavi, Hadas and Dreyfus (1994) describe a project for non-mathematically oriented high school students which stimulates sense-making, graphing, estimation, reasonableness of answers. We found the following solution (Figure 24), produced by a student learning with this approach, both surprising and elegant.

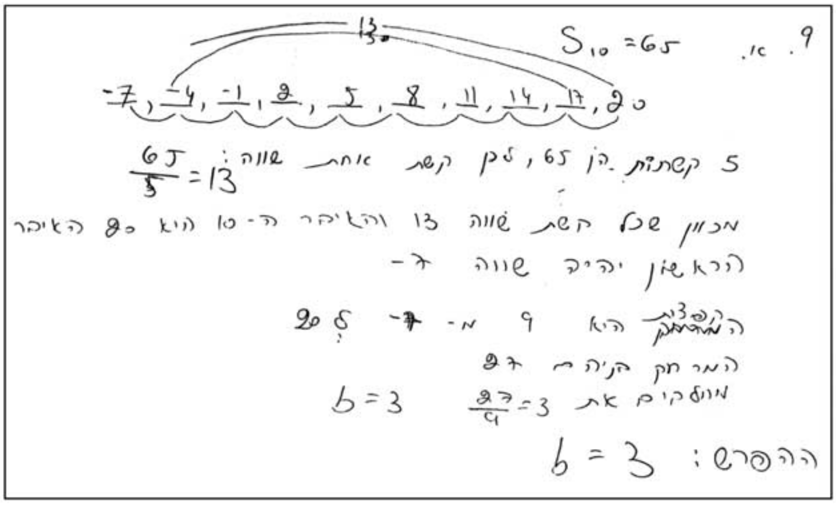

Figure 24. A visual non-symbolic solution.

Given were: a) the tenth term of an arithmetic sequence (a_{10} = *20*) and b) the sum of the first *10* terms (S_{10} = *65*). The student found the first element and the constant difference mostly relying on a visual element: arcs, which he envisioned as depicting the sum of two symmetrically situated elements in the sequence, and thus having the same value. Five such arcs add up to *65*, thus one arc is *13*. Therefore, the first element is *13–20 = –7*. Then, the student looked at another visual element: the ‘jumps’, and said that since there are *9* jumps (in a sequence of *10* elements starting at *–7* and ending at *20*), each jump must be *3*.

diSessa et al. (1991) describe a classroom experiment in which young students are encouraged to create a representation for a motion situation, and after several class periods they ended up ‘inventing’ Cartesian graphing. By being not just ‘consumers’ of visual representations, but also their collective creators, communicators and critics, these students developed meta-representational expertise, establishing and using criteria concerning the quality and adequacy of representations. Thus visualization was for them not only a way to work with pre-established products, but also was in itself the object of analysis.

When a classroom is considered as a micro-cosmos, as a community of practice, learning is no longer viewed only as instruction and exercising, but also becomes a form of participation in a disciplinary practice. It is in this context that Stevens and Hall (1998, p. 108) define 'disciplined perception'. Visualization by means of graphs, diagrams and models is a central theme which "develop and stabilize ... in interaction between people and things". Ways of seeing emerge in a social practice as it evolves.

Nemirovsky and Noble (1997) describe a research study, in which a student makes use of a physical device which served as a transitional tool used to support the development of her ability to 'see' slope vs. distance graphs.

In sum, new curricular emphases and approaches, innovative classroom practices and the understandings we develop from them, re-value visualization and its nature placing it as a central issue in mathematics education. This should not be taken to mean that visualization, no matter how illuminating the results of research, will be a panacea for the problems of mathematics education. However, understanding it better should certainly enrich our grasping of aspects of people's sense making of mathematics, and thus serve the advancement of our field.

Paraphrasing a popular song, I would suggest that 'visualization is a many splendored thing'. However, borrowing the very last sentence from (the English poet) Thomas Gray's (1716–1771) poem entitled "On the death of a favourite cat, drowned in a tub of gold fishes" (whose story can be easily imagined), I would also add

Not all that tempts your wand'ering eyes
and heedless hearts, is lawful prize;
Nor all, that glisters, gold.

NOTES

1. I would tike to thank Dr Anna Sierpinska for calling my attention to this interesting and important issue.
2. The graph only illustrates the case for b>0, but the argument obviously holds for b<o.

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