Mathematical Reasoning: Proving and Proof Production

Paola Iannone & Elena Nardi

University of East Anglia

School of Education and Lifelong Learning

United Kingdom

University of Athens

University of Cyprus

An Example

Consider the following exercise, given to Year 1 university students in mathematics:

EXERCISE 1: Write out carefully the meaning of the statement "The sequence $\{a_n\}$ converges to A as $n \to \infty$ ".

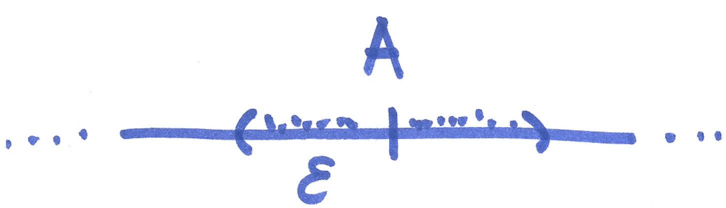
Consider now the two responses:

Response 1: $\forall\ \epsilon > 0\ \exists\ N \in \mathbf{N}\ \mid$ for $n \geq N\ \ \Rightarrow |a_n - A| < \epsilon.$

Consider now the two responses:

Response 1: $\forall\ \epsilon > 0\ \exists\ N \in \mathbf{N} \mid$ for $n \geq N \Rightarrow |a_n - A| < \epsilon$.

Response 2: The sequence $\{a_n\}$ converges to A as $n \to \infty$ if given any box of width ϵ and centre A, no matter how small I chose ϵ, from a certain point onwards all the elements of the series fall into this box.



What can we say about these responses?

The question asked was ambiguous, but both responses are correct. However, they resort to very different **types of understanding** of the convergence of a series.

What can we say about these responses?

The question asked was ambiguous, but both responses are correct. However, they resort to very different **types of understanding** of the convergence of a series.

In the remaining of this talk I will explore a theoretical framework that tries to make sense of these different types of understanding. After we have worked on some data from the project with the mathematicians that Elena introduced we will see how they, from their position of teachers and experts, make sense of these different ways of understanding (and using) mathematical concepts.

Many theoretical frameworks have tried to make sense of these different ways of understanding mathematical concepts. The most popular among mathematics educators have been:

- Instrumental and relational understanding
  (Skemp, 1976)

Many theoretical frameworks have tried to make sense of these different ways of understanding mathematical concepts. The most popular among mathematics educators have been:

- Instrumental and relational understanding (Skemp, 1976)

- Concept Image and Concept Definition (Tall & Vinner, 1981)

Syntactic and Semantic proof production

Weber and Alcock (2004) define **syntactic proof production** as

> ...one which is written solely by manipulating correctly stated definitions and other relevant facts in a logically permissible way. [...] A syntactic proof production can be colloquially defined as a proof in which all one does is unwrap the definitions and push symbols. (p210)

Syntactic and Semantic proof production

Weber and Alcock (2004) define **syntactic proof production** as

. . . one which is written solely by manipulating correctly stated definitions and other relevant facts in a logically permissible way. [. . .] A syntactic proof production can be colloquially defined as a proof in which all one does is unwrap the definitions and push symbols. (p210)

and as **semantic proof production**

. . . one in which the prover uses instantiation(s) of the mathematical object(s) to which the statement applies to suggest and guide the formal inferences that he or she draws. (p210)

Syntactic Knowledge

The abilities and knowledge required to produce syntactic proofs about a concept appear to be relatively modest. The prover would need to be able to recite the definition of a mathematical concept as well as recall important facts and theorems concerning that concept. The prover would also need to be able to derive valid inferences from the concepts definition and associated facts. We say that one who possesses these skills has a syntactic knowledge or a formal understanding of this concept.

(Weber & Alcock, 2004, p 229)

Semantic Knowledge

Semantic proof production requires the following abilities on the part of the prover:

- One should be able to instantiate relevant mathematical objects.

Semantic Knowledge

Semantic proof production requires the following abilities on the part of the prover:

- One should be able to instantiate relevant mathematical objects.
- These instantiations should be rich enough that they suggest inferences that one can draw.

Semantic Knowledge

Semantic proof production requires the following abilities on the part of the prover:

- One should be able to instantiate relevant mathematical objects.

- These instantiations should be rich enough that they suggest inferences that one can draw.

- These instantiations should be accurate reflections of the objects and concepts that they represent. That is to say, these instantiations should not suggest that the associated concepts have properties that are inconsistent with the formal theory.

Semantic Knowledge

Semantic proof production requires the following abilities on the part of the prover:

- One should be able to instantiate relevant mathematical objects.
- These instantiations should be rich enough that they suggest inferences that one can draw.
- These instantiations should be accurate reflections of the objects and concepts that they represent. That is to say, these instantiations should not suggest that the associated concepts have properties that are inconsistent with the formal theory.
- One should be able to connect the formal definition of the concept to the instantiations with which they reason.

(Adapted from Weber & Alcock, 2004, p 229)

Back to my example.

Response 1: $\forall\ \epsilon > 0\ \exists\ N \in \mathbf{N}\ \mid$ for $n \geq N\ \ |a_n - A| < \epsilon.$

This response shows that the student has mastered the syntax of this part of mathematics, or at least he has learned how to write correctly the definition required. However, we have no clear information on whether the student will be able to use this definition in a problem solving situation for example.

Response 2: The sequence $\{a_n\}$ converges to A as $n \to \infty$ If given any box of with ϵ and centre A, no matter how small I chose ϵ, from a certain point onwards all the elements of the series fall into this box.

From this response we might infer that the students has grasped the semantic meaning of convergence of a sequence. He has created a mental image s/he can work with and this is compatible with the formal definition. However there is no indication that this student will be able to use mathematics' formal language in a proof task.

The data included in the handout illustrate some of the mathematicians' ideas on these different approaches. We now take some time to read these data and work on them.

I would like you to split in groups of 3-5 and work together on the interview extracts to *analyse* the mathematicians' words in terms if the framework I have presented.