Post-Graduate Mathematics Education Summer School

Alexandroupolis, Greece

July 2009

Working Session:

Formalism, Proof and Abstraction in Upper Secondary Mathematics

TIMETABLE

Saturday Session:

Welcome – Introduction (30 minutes)

Welcome

Brief overview of research in the field of upper secondary and post-compulsory mathematics education by Elena Nardi.

A. Research Perspectives on Teachers and Teaching

A.1. "Pedagogical perspectives of mathematicians on Proof and Proving" by Paola Iannone and Elena Nardi (75 minutes)

Break (30 minutes)

A.2. "Pedagogical perspectives of secondary mathematics teachers on Proof and Proving" by Irene Biza and Elena Nardi (75 minutes)

Sunday Session:

B. Research Perspectives on Student Learning

B.1. "Learning Analysis in the context of electronic environments" by Victor Giraldo and Irene Biza (90 minutes)

Break (30 minutes)

B.2. "Natural and Formal learners" by Marcia Fusaro Pinto (60 minutes)

Overall Discussion (30 minutes)

A. Research Perspectives on Teachers and Teaching

A.1. Pedagogical Perspectives of Mathematicians on Proof and Proving

Paola Iannone, p.iannone@uea.ac.uk
Elena Nardi, e.nardi@uea.ac.uk
University of East Anglia, Norwich, UK

In what follows you will find some extracts from the interviews with the mathematicians. The participants are indicated by capital letters, the interviewers are indicated by the letters PI and EN. Before the interview data we include the examples of students' work that the mathematicians were commenting upon.

The exercise in question, included in the first year Basic Analysis and Algebra course, was:

Exercise 1: Write out carefully the meaning of the statement "The sequence a_n converges to A as $n\to\infty$.

Some of the students' written work:

Student MR

4. a) "the sequence (a_n) converges to A as $n \to \infty$"

As n approaches infinity, (a_n) is eventually arbitrarily close to A:

$$\forall \varepsilon > 0,\ \exists N,\ n > N \Rightarrow |a_n - \lambda| < \varepsilon$$

Student N

4a) $\forall \varepsilon > 0\ \exists N$: if $n \geq N$, $|a_n - A| < \varepsilon$

In the following extract the mathematicians discuss the students' work.

E: And this is difficult because we are not consistent ... I mean I try in analysis to convey the idea, if you like, that by definition the last line of the proof is the definition of something. And that view is not ... that approach is not viewed across their homework. So a question that says prove… and the majority of that questions that they see in their homework ... that does not mean that the last line of their solution is the definition, the statement that came in the second line of the statement that begun with prove. And that itself is a difficult idea and is one of the things that first year analysis should be teaching them. And on the other hand, yes, I mean it is not ... of course that is not how we do mathematics and that is not how they have done mathematics so ... it is difficult.

EN: I think my concern here is even more basic than that. Closer to what you were saying earlier, it is about: do they see that that mechanism that the definition is proposing says something about ... that very simple thing about ... that goes somewhere ... that set of numbers goes somewhere ... converges to something. ...

A: I would go a little bit further I think what ... I mean, what worries me in this is that I find it myself is that I find it very difficult to work with statement which have quantifiers over two or three quantifiers ... So the only way for myself in which I can unravel such things is that I have to build up a mental picture by which I know, ok, this is going to ... this is what is going on. I am looking for the longest of all shortest paths in a graph or for all shortest longest paths in the graph. And you see that these are horrendously difficult things. Now if I just work with these longest shortest or shortest longest, just a different example, I do not know what I am talking about, unless I have a clear picture. So when it comes to convergence I think that the primary notion for the students is asking that no matter what I specify the δ region about the a, from a certain point onward everything fits inside this box. So ... So I would say unless the student has that, has the primary way of thinking about it and it is then formalized by saying, yes, for every ε widths of the box there exists a cut off point from which onwards the whole thing sits inside ... Unless that is the direct connection between images that you have and formalization I think you are lost. If you just are juggling around ε and δ then it is a completely unworthy process. I don't think that this would ... Very unworthy attempt because it leads people into formalistic nonsense.

E: But it is very interesting. I mean ... One of the things…... one of the things I that we do is that we write these definitions many times and the sentences too. The words I use are arbitrarily and eventually. So arbitrarily and whenever we mean arbitrarily is for all ... and eventually is there exists ... And I checked some of them and they don't write down the words in the lecture notes, they write down the definitions. And they don't write down this irrelevant waffle.

D: Some of them do ...

E: I have tried many things ... I have used colored chalk sometimes for grouping the pieces of the quantified sentence. This is the bit with arbitrarily and this is the bit with eventually ... and so on. It is difficult ... it just is very hard. I mean the other striking thing is that if you talk to second and third year students, it is not done in any scientific fashion, but informally, they retrospectively understand this, to an amazing extent ... And we have relatively ... I mean, I have ... I will not name names but maybe I should ... one of the charming postgraduate tutors on this course to help his students had written out a careful sort of verbal explanation of what this means and he thought he showed it to me first and then ... it was wrong! Just wrong. And it took a long time to make him see why it was wrong. And he shouldn't be embarrassed because there has been a long correspondence a while back in the AMS notes between professional mathematicians with wrong definitions. So he said that ... the amount of what he said is that the amount that you get this means that you get closer to this number, which is just wrong. And ... it is just a hard concept, a very hard concept. And I am very keen on verbalizing it, geometrizing it and making pictures of it and as long as they stay this side of the line of correctness. And there are published analysis books with the wrong definition of convergence, just completely wrong.

A: And that is just actually ... I mean…what I wanted to say ... I would like to talk ... sort of quantifiers ... this working with intuitively geometric pictures when it comes to convergence is something that is very private: some people work like this and some don't. And I can well imagine that there are students that can work along a string of quantifiers they just do what they are told. You can view this as the recipe, you can do this, you do this and you do this ... You just follow the steps. And in some ways they are safer because there is not, you know ... they will not make mistakes as long as they are technically doing the right steps. So, you see, this depends on how you can think about mathematics and we all are different in how we can see this things and how we can work with it.

E: Sort of ... I push very, very hard number lines, that they should draw number lines, have pictures and so on ... but I also push hard, later on ... You see, no human can have a ... good intuitive geometrical or pictorial view of what the statement "the series does not converge" means, for example. I don't think ... Or say certainly no one can have a geometrical view of the statement "this function is not uniformly continuous", let's say.

A: Do you get the sense that using quantifiers as symbols is often not so productive? I mean I for instance, if at all possible, I will hardly ever use a quantifier when I write things down on the board because it kills the sentence. So I rather write down for all ... something happens if something else happens and so on, so that at the end they get the sentence in English ... Because you know ... I know, but I mean in the algebra that we ere discussing last time there are still people, students, who have disjoint bits of sort of thoughts somewhere, where it is not still clear how to go from here to there with the various implications and so on. And somehow this encourages students, you know, where you can stick things together so the students who is maybe not clear what he has written down can get away hoping that somebody will decipher it correctly. This is the thing that students think to do ...

E: Yes

A: While if it was a sentence in English there is more encouragement to …

E: Yes, I guess ... I mean in something the power of symbols is something that they learn to manipulate because to encourage them more algorithmic things. So for example the negation of a quantified statement I think is much easier as symbolic definition. Because it is an algorithm: you replace for all with the exists, you replace the exists with for all and then you get a statement and that is the algorithm. Yes…

A: Exactly ... I am interested in the evidence from the students ...

E: But there is always also this poor looking thing that somehow… is not automatically what they are most comfortable with, what pushes them further towards what next they need to do. And… It is stunning to see ... A sequence converges when ... And one of the things that I would give as aspiration for the course is to learn to view that ... to view that as the same somehow ... is a fluency that is required ...

EN: Yes I think that this interplay between the words and the symbols that is in your suggested solution you are asking ... I think is something that, you know, you want the students to aspire to and achieve at some point ... I mean, somehow they have to see the symbols as compact compressed, part of this occurrence of meanings, that this is actually a useful thing to have, not an obstacle to understand. That is a problem also with certain groups of this example that they somehow ... they see this as a burden of mathematics, not actually as a useful part ...

E: I think this is a wide problem particularly in pure maths. The process of pure maths as it proceeds invests more and more meaning in purer and purer symbols and so you end up with this capital *K* subgroup stuff, let's say. And, you know, you can literally write down a one-line statement that would take ten years to explain. And that is very unfamiliar for them. They think that a calculation that is twice as hard is much longer and the integral is twice as difficult if it has twice many steps in it. And a differential equation that is twice as difficult has twice as many steps in it. And ... whereas this is just not as pure mathematics is written.

A.2. Pedagogical Perspectives of Secondary Mathematics Teachers on Proof and Proving

Irene Biza, i.biza@uea.ac.uk
Elena Nardi, e.nardi@uea.ac.uk
University of East Anglia, Norwich, UK

The tangent task:

Σε μαθητές κατεύθυνσης της Γ΄ Λυκείου δόθηκε η ακόλουθη άσκηση:

‘Να εξετάσετε αν η ευθεία με εξίσωση $y = 2$ είναι εφαπτομένη της γραφικής παράστασής της συνάρτησης με τύπο $f(x) = 3x^3 + 2$.

Δύο μαθητές έδωσαν τις παρακάτω απαντήσεις.

Μαθητής Α:

‘Θα βρω τα κοινά σημεία της ευθείας και της γραφικής παράστασης λύνοντας το σύστημα:

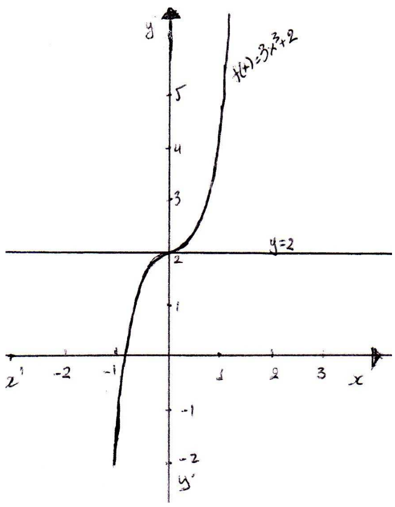
$$\begin{cases} y = 3x^3 + 2 \\ y = 2 \end{cases} \Leftrightarrow \begin{cases} 3x^3 + 2 = 2 \\ y = 2 \end{cases} \Leftrightarrow \begin{cases} 3x^3 = 0 \\ y = 2 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = 2 \end{cases}$$

Το κοινό σημείο είναι το $A(0,2)$.

Η ευθεία είναι εφαπτομένη της γραφικής παράστασης στο σημείο A γιατί έχουν ένα μόνο κοινό σημείο (το A).’

Μαθητής Β:

‘Δεν είναι η ευθεία εφαπτομένη της γραφικής παράστασης γιατί παρόλο που έχει ένα κοινό σημείο τη διαπερνά στο σημείο αυτό, όπως φαίνεται στο σχήμα’



α) Ποιος θεωρείτε ότι είναι ο στόχος της παραπάνω άσκησης;

β) Πώς ερμηνεύετε τις επιλογές που έκανε ο καθένας από τους παραπάνω μαθητές στην απάντηση του.

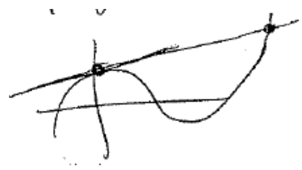
γ) Τι σχόλια θα κάνατε σε κάθε ένα από τους παραπάνω μαθητές σχετικά με την απάντηση που έδωσε;

ΓΡΑΠΤΕΣ ΑΠΑΝΤΗΣΕΙΣ[1] ΤΟΥ ΣΠΥΡΟΥ ΚΑΙ ΤΗΣ ΑΝΝΑΣ

ΣΠΥΡΟΣ

α. Στόχος είναι η αποσαφήνιση της έννοιας της εφαπτομένης γραφικής παράστασης και του μαθηματικού ορισμού της.

β. Ο πρώτος μαθητής συγχέει την έννοια της εφαπτομένης σε γραφική παράσταση, με την έννοια της εφαπτομένης σε κύκλο (ή κωνική τομή γενικά). Δηλαδή «ένα μόνο κοινό σημείο». Αλλά του εξηγούμε:



Ο δεύτερος μαθητής συγχέει το «διαισθητικό» ορισμό της εφαπτομένης («γλύφει» τη γραφική παράσταση) με το μαθηματικό ορισμό της. (Ωστόσο μερικές φορές τα Μαθηματικά έρχονται σε αντίθεση με τη διαίσθηση π.χ. $\sqrt{2}$, i κλπ.)
Σχόλια: Και οι δύο μαθητές δεν έχουν αφομοιώσει τον μαθηματικό ορισμό της εφαπτομένης σε γραφική παράσταση. Αντίθετα έχει ισχυροποιηθεί μέσα τους μια λανθασμένη αντίληψη.
Χρειάζεται λοιπόν να ανατρέξουμε στους λόγους που οδήγησαν στον σύγχρονο ορισμό (πχ. από τη Φυσική κλπ.), και να δώσουμε πολλά παραδείγματα και αντιπαραδείγματα.
Τελικά θα πρέπει να οδηγούμε τους μαθητές στο να δεκτούν και να κατανοήσουν το μαθηματικό ορισμό.

ΑΝΝΑ

α. Ο στόχος της παραπάνω άσκησης είναι το να βοηθήσει τους μαθητές να κατανοήσουν ότι η εφαπτόμενη της γραφικής παράστασης μιας συνάρτησης δεν είναι οποιαδήποτε ευθεία έχει 1 κοινό σημείο με τη γραφική παράσταση της συνάρτησης. Με τη συγκεκριμένη άσκηση οι μαθητές θα κατανοήσουν καλύτερα την έννοια της εφαπτόμενης.

β. Θεωρώ ότι ο Α μαθητής έχει κατανοήσει λάθος την έννοια της εφαπτόμενης. Επίσης πολύ πιθανό είναι να ~~θεωρεί ότι βρίσκοντας~~ μη σκέφτηκε ότι κοινό σημείο με μια συνάρτηση δεν έχει μόνο η εφαπτόμενη της στο συγκεκριμένο σημείο αλλά άπειρες άλλες ευθείες, αφού από ένα σημείο διέρχονται άπειρες ευθείες.
Ο Β μαθητής επέλεξε να φτιάξει τη γραφική παράσταση, αν και συνήθως τα παιδιά την αποφεύγουν. Θεωρώ ότι είναι σωστή η λύση του.

γ. Στον Α μαθητή θα του εξηγούσα με τη βοήθεια της γραφικής παράστασης ότι αν μια ευθεία έχει 1 κοινό σημείο με τη γραφική παράσταση της συνάρτησης δε σημαίνει ότι είναι και εφαπτόμενη της. Επίσης θα του ξαναέδειχνα τον τρόπο εύρεσης εφαπτόμενης με τον τύπο $y\text{-}f(x_0)=f'(x_0)(x\text{-}x_0)$. Και θα του έδινα ένα φύλλο εργασίας πάνω στις εφαπτόμενες ~~όπου θα του ζητούσα~~.
Στο Β μαθητή θα του έλεγα ότι είναι σωστή η απάντηση του, αλλά τον ρώταγα πως θα έλυνε την άσκηση αν δινόταν μια συνάρτηση την οποία θα ήταν δύσκολο να κάνει τη γραφική παράσταση. Γι' αυτό το λόγο θα του έδειχνα τον τρόπο εύρεσης εφαπτομένης με τον τύπο $y\text{-}f(x_0)=f'(x_0)(x\text{-}x_0)$. και θα του έδινα ένα φύλλο εργασίας.

[1] Για λόγους ανωνυμίας δεν παραθέτουμε τα αυθεντικά αντίγραφα των γραπτών αλλά πιστή αντιγραφή των απαντήσεων. Καμία δική μας προσθήκη δεν έχει γίνει. Τα ονόματα των συμμετεχόντων είναι ψευδώνυμα.

ΑΠΟΣΠΑΣΜΑΤΑ ΑΠΟ ΤΙΣ ΣΥΝΕΝΤΕΥΞΕΙΣ ΤΟΥ ΣΠΥΡΟΥ ΚΑΙ ΤΗΣ ΑΝΝΑΣ

ΣΠΥΡΟΣ

Ε: Γενικά, μια απόδειξη που θα στηριζότανε στο σχήμα θα τη δεχόσασταν ή οχι;
Σ: Όχι. Δεν τη δέχεται το εξεταστικό σύστημα καταρχήν.
Ε: Μαθηματικά θα ήταν αποδεκτή; Ας αφήσουμε τις εξετάσεις.
Σ: Μαθηματικά μέσα στη τάξη θα την καλωσόριζα σε επίπεδο μαθήματος και θα την ανέλυα και θα την αναδείκνυα, αλλά όχι σε ένα διαγώνισμα.
Ε: Όταν λέτε θα την καλωσόριζα μέχρι ποιο σημείο; Δηλαδή, θα σας αρκούσε αυτή ή θα ζητάγατε κάτι ακόμα;
Σ: Θα προσπαθούσα να οδηγήσω την κουβέντα στην κανονική απόδειξη. Μέσα από αυτή θα προσπαθούσα να οδηγήσω την κουβέντα στην κανονική απόδειξη.
Ε: Όταν λέτε κανονική…
Σ: Με τον ορισμό, με την κλίση με την παράγωγο κλπ.
Ε: Γιατί; Δεν θα ήταν αρκετή αυτή;
Σ: Έτσι έχουμε μάθει νομίζω οι μαθηματικοί μέχρι τώρα. Να ζητάμε την ακρίβεια. Την αξιωματική, έχουμε αυτή την αξιωματική αρχή στο μυαλό μας. Ότι λέω το αποδεικνύω με βάση τα αξιώματα, με βάση τα θεωρήματα, με βάση…Και αυτό ζητείται και στις εξετάσεις. Και υποτίθεται ότι προετοιμάζουμε τα παιδιά για τις εξετάσεις.

ΑΝΝΑ

Ε. […] ας υποθέσουμε ότι ο δεύτερος έλεγε ότι είναι εφαπτομένη έκανε το σχήμα αυτό και έλεγε ότι είναι εφαπτομένη, τότε θα τη δεχόσουνα αυτή την απάντηση; […]
Α. Ότι είναι τώρα … μμμ … βασικά νομίζω όμως ότι πρέπει να κάνουμε όλη τη διαδικασία τη προηγούμενη [ως προηγούμενη διαδικασία θεωρούμε την αλγεβρική]
Ε. Γιατί;
Α. Γιατί … γιατί μπορεί αυτή η ευθεία να ήταν και εδώ [δείχνει στο χαρτί] … να ήταν πιο πάνω πιο κάτω που δεν είναι εφαπτομένη.
Ε. Ναι … ναι …
Α. Δηλαδή το ότι περνάει και διέρχ … κόβει στη μέση τη γραφική παράσταση μόλις το βλέπω δε μπορώ να αποκλείσω ότι δεν είναι εφαπτομένη αλλά δε μπορώ να πω ότι είναι, κιόλας … δε πρέπει να κάνω κάποιες …
Ε. ναι ναι ναι … τότε όμως το δέχτηκες […]
Α. ναι … το δέχτηκα γιατί έλεγε ότι δεν είναι [το 'δεν είναι' τονίζεται στο λόγο της] και εγώ είχα στο μυαλό μου ότι με το που βλέπω ότι περνάει δεν υπάρχει περίπτωση, ότι και να ήτανε
Ε. Γενικά θα δεχόσουνα μία γραφική λύση … αν με κάποιο τρόπο … πίστευες ότι είναι σωστή η απάντηση; […]
Α. Αν ήτανε σωστή θα τη δεχόμουνα
[…]
δηλαδή δεν είναι απαραίτητο να πηγαίνουμε με την αλγεβρική μέθοδο με τύπους και αυτά, αυτό πιστεύω δε ξέρω αν είναι σωστό [γελάει ελαφρά] […]
Α. Απλά πιστεύω ότι οι μαθητές δεν έχουν τόσο εξοικειωθεί με τις γραφικές παραστάσεις … και για αυτούς είναι πιο εύκολο να χρησιμοποιούν τύπους … το παίρνουν πιο πολύ σαν μεθοδολογία σαν … πιστεύω ότι δεν έχουν μπει σε βάθος για να ξέρουν να φτιάχνουν παραστάσεις τέλεια και να ξέρουν να τις ερμηνεύουν και τόσο καλά και για αυτό συνήθως οι περισσότεροι χρησιμοποιούν αλγεβρικούς τρόπους […] γιατί για να κάνει τη γραφική παράσταση και να την αναλύσεις πρέπει να έχει καταλάβει κάτι πάρα πολύ καλά … να το έχεις κάνει κτήμα σου, τελείως, ενώ αυτό, μαθαίνεις ένα τυφλοσούρτη κάπως και το λύνεις, εγώ αυτό θέλω […] Πάντως αν ήταν σωστό θα το δεχόμουνα γιατί θα καταλάβαινα ότι το έχει καταλάβει

πιο καλά από κάποιον που μπορεί να ακολουθήσει τους αλγεβρικούς τύπους … τώρα δε ξέρω, έχω δίκιο?

[Αργότερα, όταν κλονίζεται με κάποια παραδείγματα η πίστη της στο σχήμα:]

A. Απλά πιστεύω ότι αφού λύσουμε με τους αλγεβρικούς τύπους και βρούμε το αποτέλεσμα, μετά καλό είναι να λέμε και στους μαθητές να κάνουν και τη γραφική παράσταση γιατί καμιά φορά φτάνουν στο τέλος λένε «α εντάξει το βρήκα» χωρίς να έχουν συνειδητοποιήσει έστω στο μυαλό τους πως θα είναι περίπου και μόλις βλέπουν έτσι μια γραφική παράσταση δε μπορούν κατευθείαν να απαντήσουν και εγώ πιστεύω ότι το ίδιο έπαθα δηλαδή είχα συνηθίσει να βλέπω εφαπτομένες σε κύκλους και είχε περάσει στο μυαλό μου υποσυνείδητα ότι όλα έτσι πρέπει να είναι … όλες οι εφαπτομένες έτσι πρέπει να είναι γιατί δεν είχα εξοικειωθεί με γραφικές παραστάσεις

B. Research Perspectives on Student Learning

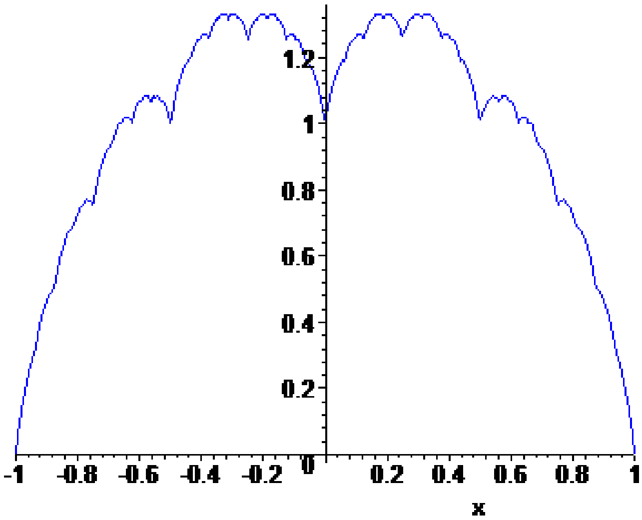
B.1. Learning Analysis in the Context of Electronic Environments

Victor Giraldo, victor@im.ufrj.br
Federal University of Rio de Janeiro, Brazil
Irene Biza, i.biza@uea.ac.uk
University of East Anglia, Norwich, UK

Part I: Descriptions and computational conflicts: The case of the derivative

Interview T3

Participants were shown the graph of the Blancmange function sketched by software Maple V. They were asked to explain in their own words the behavior of the function. They were familiar with the function and its graph sketched by the software, as this had been discussed in a previous lecture in the computer lab. They were free to use to computer, as well as pencil and paper, at any moment of the interview.
In the transcripts below, *ellipsis* (...) represents moments of pause and *exclamation mark* (!) represents moments of excitement.



Note: The Blancmange function was proposed by Teiji Takagi in 1903. It is defined as the sum of the infinite series:

$$b:\mathbb{R}\to\mathbb{R} \qquad b(x)=\sum_{n=0}^{+\infty}\frac{f(2^n x)}{2^n}$$

Where:

$$f:\mathbb{R}\to\mathbb{R} \qquad f(x)=\inf_{n\in\mathbb{Z}}|x-n|$$

Thus, the Blancmange function is the uniform limit of continuous functions (sinse the *n*-th term has modulus not greater than $\frac{1}{2^n}$). Therefore, it is continuous. On the other hand, the *n*-th term of the infinite sum is non differentiable at the points of the form $\frac{k}{2^{n+1}}$, with $k\in\mathbf{Z}$. It can be proved that it is nowhere differentiable.

Episode *A,* Tiago

Tiago: *We took several modulus functions and add them up (...) In order to make this function to have (...) kind of (...) The first one has one corner, the second would have, the third will have four and so on and so forth. Then we noticed it's* 2^n*, that increases the quantity of corners. And as we want* n *tending to infinite, this function [*points to the blancmange on the screen*] would be (...) every point would be a corner. (...) But to think of a corner one must have a part of the function to be kind of (...) continuous like this [*waves hand showing a straight line*] at least in a tiny interval. In this function, it's weird because that is not such interval.*

Researcher: *Please, explain to me better what you think of as so weird.*

Tiago: *It's (…) in order for a corner to exist, it must have (…) sides. And sides are basically straight lines or something like this. But all the points in the function are corners, so there's no room for sides or anything else. So, how can this function have infinite corners if it cannot have any sides for them?*

Researcher: *And how do you think about this property, the fact these sides don't exist?*

Tiago: *It's very hard to imagine. One thing that has a corner, but doesn't have the characteristic of a corner, that is, tiny straight lines or spaces defining that corner.*

Researcher: *What do you expect to see when you zoom in on this?*

Tiago: *I've got no idea. I cannot see that in my mind.*

Researcher: *But you just said that these corners must have sides of some sort.*

Tiago: *Yes, they must have exactly two (…) segments to form these corners. That's the weirdest thing!*

Researcher: *You said something about this sides being continuous. What did you mean by that?*

Tiago: *No, I meant little straight segments. I don't mean continuity, I used the wrong word. I understand it's all continuous. (…) It'd be a segment, two segments to form this corner. And in this case, these segments don't exist, but the corners do.*

Researcher: *Would you like to zoom in on this?*

Tiago: *Yes, we can do it.*

Researcher: *You said you had no idea what you'd see when zooming in. Aren't you curious?*

Tiago: *Yes, I don't understand and I'd like to. But I don't think the computer would help.*

Researcher: *Why not?*

Tiago: *Well, it always does things wrong.*

Researcher: *All right, but let's try though.*

Tiago: *Right. [*Zooms in*] But theses are (…) straight lines. Why are there straight lines?*

Researcher: *What do you think?*

Tiago: *[*After thinking for a few minutes*] Actually, I think (…) Well, that cannot be really the Blancmange, can it?*

Researcher: *Why not?*

Tiago: *Because if it was, we'd have to see more and more (…) corners (…) it'd be more and more (…) a weird behavior, wrinkled. But it doesn't, so I conclude it's the function! It's an optical illusion, an approximation. Now I realise! The computer couldn't make it anyway. Of course, it cannot be it!*

Researcher: *Why not?*

Tiago: *Because it's the result of adding infinite things. The computer cannot add infinite things. It doesn't even know what that means.*

Researcher: *So, in this case what's that?*

Tiago: *It's one of the sums in the middle, one approximation, isn't it?*

Researcher: *Exactly. You see here [*points to the screen*]. We add until the tenth function.*

Tiago: *I see. I think that in this case, we couldn't speak in terms of corners.*

Researcher: *Carry on.*

Tiago: *It's hard to put in words, but I guess I understand it better.*

Researcher: *All right, but try.*

Tiago: *Well, the computer can make the sums up to 10, to 100, or whatever. But it cannot make it to infinite. It can sketch all these sums, to 10, to 100, like this, each you if we want. (...) So, these are made of corners. (...) But this one here (...) It's not differentiable, but it's from another kind (...) another kind of thing, of aspect. We cannot speak of corners, like in the modulus function. This one would be a more abstract thing. We couldn't sketch it, nor even imagine it!*

Researcher: *And why do you think you understand it better now?*

Tiago: *Well, I cannot sketch and (...) that's it, full stop! That's not sketch-able. It's not corners, but another thing. I thought non differentiability were corners (...) a synonym of corners. Now I see (...) in order to understand this I must think in a different way, not through the corners, not through the sketching. Non differentiability is not (...) I mean (...) not only corners!*

Episode *B*, Antônio

Antônio: *Well, I didn't understand.*

Researcher: *What?*

Antônio: *I understood the following. You took the modular functions, then reflected them upwards, right? This way you kept the curve above the Ox axis. Then you did the following, you took the same curve and divided it by 2, or multiplied by ½, and reflected upwards. After that, you took this same one and divided by 2 and reflected upwards. That is, you multiplied by (...) as if you had taken the first one and had multiplied by ¼. So, if you notice, if it is continuous this thing you're doing, it is as if you were taking a number and multiplying it by ½, taking what's been multiplied and multiply by ½, by ½. So, it's a geometric progression of ratio ½. Do you understand? So, the graph is the summation of these initial curves. So it's the sum of a geometric progression. The sum of a geometric progression is a limit. So, it converges to a point. What I observe here is the following. In my opinion, each point of the curve (...) those little triangles, they converge to each point of the curve. The curve is like (...) the union of a sum of geometric progressions.*

Researcher: *All right, but you understand that, don't you?*

Antônio: *I do, very clearly.*

Researcher: *So, what don't you understand?*

Antônio: *How can this sum be linear.*

Researcher: *Linear?*

Antônio: *It should be wrinkled.*

Researcher: *This looks linear?*

Antônio: *More or less. It should be more wrinkled. And when you zoom in, it gets linear. It should get more wrinkled.*

Researcher: *But how do you know that?*

Antônio: *Because I've done it, at home. I've zoomed in. And it became linear. It doesn't look really wrinkle here [*points to the screen*] but it should get more broken, wrinkled. It does not.*

Researcher: *Ah. And why is that, in you opinion?*

Antônio: *Well, you know. The computer just cannot solve it.*

Researcher: *So what is the solution?*

Antônio: *It's the sum of a geometric progression, the infinite sum. But obviously that's not what the computer gives us.*

Researcher: *So what is it that the computer gives to us?*

Antônio: *Man, this is what I want you to explain to me!*

Researcher: *All right. This is just the sum of the geometric progression until 10. Do you see here?*

Antônio: *Ah!! Of course, that's right man! All right, all right!*

Researcher: *We could carry on with the sum.*

Antônio: *Yes, much more. But infinity it [*the computer*] cannot make. (...) The computer, or any other thing!*

Researcher: *Why not?*

Antônio: Because (...) one cannot add anything up to infinite! There will always remain an infinity missing!!! *And nothing can represent the infinite as a whole. But only show that it tends to that place, to that thing, but not that it actually is.*

Researcher: *Carry on.*

Antônio: *That's inifinity, man. You cannot represent if, not in the plane, not in the space, not in any **R.***

Researcher: *And in the computer?*

Antônio: *In anything! The computer represents what the man knows.*

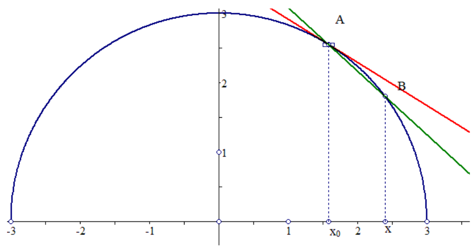
Researcher: *Do you want to say something else?*

Antônio: *Yes. Man, I've tried to grasp this curve, you know. I know the following. I thought, it's not formed by straight lines, not even locally. Because then it'd be differentiable. So, it's not formed by straight lines, segments, let's say so, points that you join in together to be tiny straight line segments. No, not even in a tiny piece. But it has points. As I said before, it's a union of points, one glued to the other. But we cannot say they're aligned, but close together in a irregular way. Then, I thought, imagine that little device, the earthquake machine. The points have nothing to do one with the other whatsoever! So, what happens? These points, so close together, but they don't make a straight line among them, you know? But it is hard to think about that. Because we must also consider the assumption that there are infinite points. So, between two points there will always be another one. So this one here (…) this graph (…) it contradicts our human intuition!*

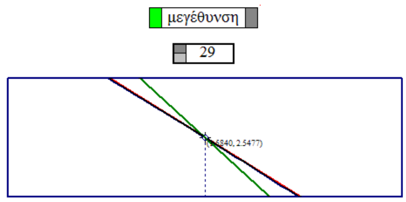
Part II: Constructing the Definition of Tangent Line

Περιστατικό 1:

Το περιστατικό αυτό αναφέρεται στο Στάδιο 2 όπου οι μαθητές έχουν ήδη δει στο ηλεκτρονικό περιβάλλον την εφαπτομένη του κύκλου ως την οριακή θέση της τέμνουσας και έχουν παρατηρήσει στο παράθυρο μεγέθυνσης ότι αυτή φαίνεται να ταυτίζεται με τον κύκλο όταν εστιάσουμε πολύ κοντά στο σημείο επαφής. Η τάξη διαπραγματεύεται την εφαπτομένη του ημικυκλίου (ως γραφική παράσταση συνάρτησης, Εικόνα 1α) και οι μαθητές καλούνται να σχολιάσουν το σχήμα που προβάλλεται στο παράθυρο μεγέθυνσης (Εικόνα 1β). Τότε, ένας μαθητής λέει: «φαίνεται να ταυτίζεται γιατί όταν πλησιάζουμε με μεγάλο αριθμό μεγέθυνσης δεν είναι ορατή η διαφορά λόγω της χαμηλής ανάλυσης της οθόνης».



Εικόνα 1α: Εφαπτομένη ημικυκλίου



Εικόνα 1β: Μεγέθυνση του κύκλου

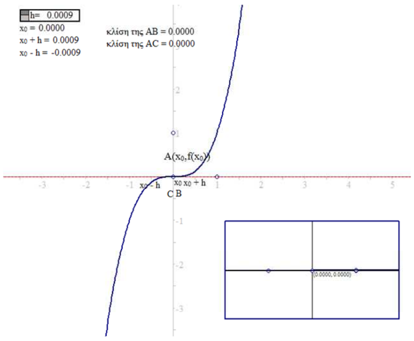
Ερωτήματα:

Α. Πώς ερμηνεύετε την αντίδραση του μαθητή;

Β. Με ποιο τρόπο θα αντιμετωπίζατε μία τέτοια αντίδραση;

Περιστατικό 2:

Στο Στάδιο 4, οι μαθητές έχουν ήδη εισαχθεί στην έννοια της παραγώγου σε σημείο και έχει γίνει η σύνδεση της με την κλίση της εφαπτομένης. Η τάξη διαπραγματεύεται ειδικές περιπτώσεις εφαπτομένης γραφικής παράστασης συνάρτησης που η προηγούμενη έρευνα έχει δείξει ως ιδιαίτερα προβληματικές στην κατανόησή τους όπως η εφαπτομένη της γραφικής παράστασης της συνάρτησης $f(x)=x^3$ στο σημείο καμπής της $O(0,0)$. Η τάξη διαπραγματεύεται την περίπτωση που φαίνεται στην Εικόνα 2 όπου η οριακή θέση των τεμνουσών, άρα η υποψήφια εφαπτομένη στο σημείο $O(0,0)$, είναι ο άξονας *x΄x*, οι κλίσεις των δύο τεμνουσών συγκλίνουν και η καμπύλη στο παράθυρο μεγέθυνσης φαίνεται να ταυτίζεται με τον άξονα x΄x.



Εικόνα 2: Εφαπτομένη σε σημείο καμπής

Καλούμε τους μαθητές να εξετάσουν αν η καμπύλη έχει εφαπτομένη στο σημείο $O(0,0)$ και ακολουθεί ο παρακάτω διάλογος μεταξύ της ερευνήτριας (Ε), των μαθητών Μ1, Μ2 και της μαθήτριας Μ3:

Μ1: Φαίνεται να είναι αλλά από την άλλη μεριά δεν μπορεί να είναι.

Ε: Γιατί δεν μπορεί να είναι;

Μ1: Δεν μπορεί να είναι εφαπτομένη γιατί τέμνει τη γραφική παράσταση … πώς να το πω … τη διαπερνά … δηλαδή τη χωρίζει σε δύο τμήματα που το ένα είναι από τη μία μεριά και το άλλο από την άλλη … από την άλλη μεριά φαίνεται να ταυτίζεται κοντά στο Α στη μεγέθυνση. Δεν ξέρω …

Ε: Και σε τι μας πειράζει που την τέμνει;

Μ2: Δηλαδή μπορεί και να την τέμνει, να τη διαπερνά;

Ε: Γιατί να μην μπορεί; Βάλαμε εμείς κάποιο τέτοιο περιορισμό στο ορισμό που δώσαμε;

Μ3: Δηλαδή οι ιδιότητες που ξέρουμε από τον κύκλο δεν ισχύουν στην περίπτωση αυτή;

Ερωτήματα:

Α. Πώς ερμηνεύετε την αντίδραση των μαθητών;

Β. Συγκριτικά με το προηγούμενο περιστατικό παρατηρείτε κάποιες διαφορές ή ομοιότητες;

B.2. Natural and Formal Learners

Márcia Maria Fusaro Pinto, [***fusaromarcia@yahoo.com.br***](mailto:fusaromarcia@yahoo.com.br)
Universidade Federal de Minas Gerais-Brasil

Introduction

Here, I am introducing you four students studying Real Analysis, named A, B, C and D. At the time, *all of them presented a visual representation of their personal concept definition of limit of a sequence*. Two of them were able to reproduce the formal definition (with minor slips). Below are brief excerpts of the interviews, where they evoke images related to the formal concept of limit of a sequence, and aspects of its use.

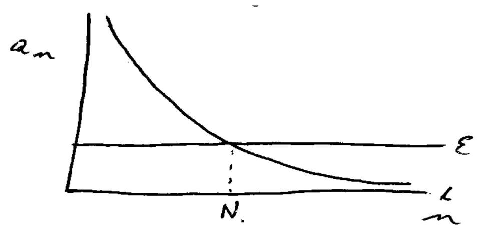
Students' evoked concept image of the formal definition

The following excerpts refer to students' comments on the meaning of the formal definition of limit of a sequence.

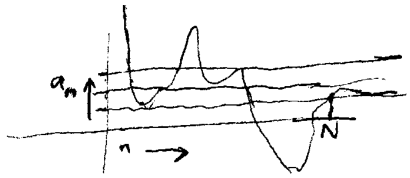
Discussion: for each case, give your ideas on the student's use of visual images and pictures in their evoked concept image.

> "I mean, the main thing ... I could understand it said: by finding the value of capital en ... it actually ... is the proof that the ... sequence is tending to zero. That was a bit awkward, I couldn't get that to start with. I could understand what the definition was saying, but, to me, it didn't seem to be proving ... that it went to zero. ... I'm not ... not fully happy with this. There's still things I've got to get sorted out. But ... I get ... I have made some progress since I started… …you just … you have got to put the values in the variables of the formula."
>
> (A, first interview)

> "Well, before I saw anyone draw that [a visual representation for the limit of a sequence he reproduced] it was just umm…thinking basically as en gets larger than capital en, a en is going to get closer to el so that the difference between them is going to come very small and basically whatever value you try to make it smaller than, if you go far enough out then the gap between them is going to be smaller. That´s what I thought before seeing the diagrams and something like that."
>
> (B, first interview)



> "…umm, [I] sort of imagine the curve just coming down like this and dipping below this point which is epsilon … and this would be en. [explaining while drawing on the left] So as soon as they dip below this point then … the terms bigger than this [pointing from *N* to the right] tend to a certain limit, if you make this small enough [pointing to the value of ε]."
>
> (C, first interview)



"I think of it graphically … I think of it as you got a graph there and the function there [while drawing the picture below], and I think that it's…it's got the limit there …and then epsilon once like that, and you can draw along and then all the … points after capital en they are inside those bounds …if this err when I first thought of this, it was hard to understand, so I thought of it like this …like that's the en going across there and that's a en …err this shouldn't really be a graph, it should be points."

(D, first interview)

On the use of the formal theory

Discussion: in each case, give your ideas on the student´s use of visual images and pictures and the formal definition when solving the following question from a worksheet.

Question from a worksheet:

If $a_n \to 1$, *prove that there exists* $N \in IN$ *such that* $a_n > \frac{3}{4}$ *for all* $n > N$

"I had no idea where to start, really. You see, the way the question was set out, I wasn´t really prepared for that. There were some examples in an example sheet, but I haven´t had time to go through them with my tutor umm I couldn´t do it at all … Because I was used to this saying a en equals … [specifying or giving a particular sequence] and add the definition of what a en was. And then there wasn´t that there, so I had nothing to start on, really …"

(A, second interview)

"It seemed to be a silly question that…if a en tends to one then if you question when a en is greater than three quarters… this is a bound, it seems … I don´t know why … I wrote something in this question … I think it was just epsilon to be … a quarter… or something, I put minus one less than a quarter … I think I worked it out written down … I don´t know whether it´s right or not because I haven´t received the sheet back …"

(B, second interview)

Student C commented he solved the exercise as it was done in classroom, applying the reverse triangle inequality.

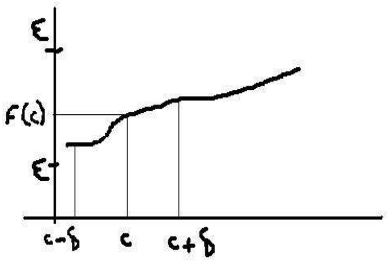
"I chose epsilon as 0.1 ... and showed that a_n lies between 0.9 and 1.1; so it must be greater than $\frac{3}{4}$."

(D, second interview)

Further analysis: preferences on the meaning and use of definitions

Two definitions of continuity of a function where given to students. Here, they declare their preferences for the $\varepsilon - \delta$ definition of continuity or the sequential definition.
Discussion: in each case, give your ideas on similarities and differences on students argumentation.

Uneasy with both definitions; explaining the meaning of one of them

	Umm … you´ve got a function and … you pick a point there …and if you say if…delta …delta…err a gap space either side of c…it goes to the function …and …do epsilon..ef of ce there… [while drawing]… and that´s means the function will, be within epsilon of ef of ce, say there [drawing] …but making sure you pick an adequate delta that fits the epsilon." (A, fourth interview)

Preference for the sequential definition of continuity

"…you are looking …you are trying to evaluate umm …umm…if there is no holes in the graph, then, see, you have got a function like that [drawing a graph of a continuous line, in a coordinate axis] ; then there will be a point there [marking a point c in the graph drawn] and umm…then umm…there is no hole at that point, then…there will be a value of ef at ce and you are getting closer and closer to ce umm and umm the closer you get to ce then the closer you get to ef of ce and you can make the difference between umm x and the value you chose at ce then … you can make that as small as you want, then …you can approximate of ef of ce as close as you want…"

(B, fourth interview)

Preferences on the $\varepsilon - \delta$ definition of continuity

"I think of the $\varepsilon - \delta$ definition was a bit more…confusing first of all, but in some ways I find it closer to … sort of convergence of a sequence we had last term … because instead of choosing umm capital en, so when you are like …you run down the sequence a en … you are choosing a delta, which is when you decide the values …of…the distance delta from ce …then the epsilon bit is true …so it´s the same sort of …then you choose the epsilon first of all, delta in this case … so when it satisfies that …it satisfies that less than epsilon."

(C, fourth interview)

Preferences on the $\varepsilon - \delta$ definition of continuity
"Yes [I prefer the $\varepsilon - \delta$ definition] umm I don´t know. It looks like …thinking of it graphically, it makes it very similar to the limit definition … where you´ve got like a bound in the graph , so all points will like be in that bound."

(D, fourth interview)

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