Mathematicians' perspectives on Proving and Proof Production

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Methods

The interview data from the mathematicians were analysed using *thematic analysis*. The Episodes considered were the ones related to proof and proving. The mathematicians' utterances were coded through the chosen theoretical framework and a commentary was written to the relevant utterances.

In the next slide I include an example of a short part of the raw data with the commentary.

UTTERANCES	COMMENTARY
A: I would go a little bit further I think what … I mean, what worries me in this is that I find it myself is that I find it very difficult to work with statement which have quantifiers over two or three quantifiers… So the only way for myself in which I can unravel such things is that I have to build up a mental picture by which I know, ok, this is going to… this is what is going on. I am looking for the longest of all shortest paths in a graph or for all shortest longest paths in the graph. And you see that these are horrendously difficult things. Now if I just work with these longest shortest or shortest longest, just a different example, I do not know what I am talking about, unless I have a clear picture. So when it comes to convergence I think that the primary notion for the students is asking that no matter what I specify the ε region about the a, from a certain point onward everything fits inside this box. So…	Difficulties in plowing through statements with quantifiers. need for a mental picture to make sense of it – semantic Informal idea of convergence
EN: And how do you react to…	
A: Let me just finish this. So I would say unless the student has that, has the primary way of thinking about it and it is then formalized by saying, yes, for every ε widths of the box there exists a cut off point from which onwards the whole thing sits inside… Unless that is the direct connection between images that you have and formalization I think you are lost. If you just are juggling around ε and δ then it is a completely unworthy process. I don't think that this would…	So first comes the primary idea: the box, then it is formalized in epsilon delta definition
EN: That is exactly my concern actually…	
A: Very unworthy attempt because it leads people into formalistic nonsense.	Formalistic nonsense!
TW: But it is very interesting. I mean…One of the things… one of the things I that we do is that we write these definitions many times and the sentences too. The words I use are arbitrarily and eventually. So arbitrarily and whenever we mean arbitrarily is for all … and eventually is there exists … And I checked some of them and they don't write down the words in the lecture notes, they write down the definitions. And they don't write down this irrelevant waffle,	E's informal definition – again with arbitrarily and eventually
D: Some of them do…	
E: I have tried many things… I have used colored chalk sometimes for grouping the pieces of the quantified sentence. This is the bit with arbitrarily and this is the bit with eventually… and so on. It is difficult… it just is very hard. I mean the other striking thing is that if you talk to second and third year students, it is not done in any scientific fashion, but informally, they retrospectively understand this, to an amazing extent…. And we have relatively… I mean, I have… I will not name names but maybe I should… one of the charming postgraduate tutors on this course to help his students had written out a careful sort of verbal explanation of what this means and he thought he showed it to me first and then … it was wrong! Just wrong. And it took a long time to make him see why it was wrong. And he shouldn't be embarrassed because there has been a long correspondence a while back in the AMS notes between professional mathematicians with wrong definitions. So he said that… the amount of what he said is that the amount that you get this means that you get closer to this number, which is just wrong. And… it is just a hard concept, a very hard concept. And I am very keen on verbalizing it,	It is near impossible to express the definition of convergence correctly in words.

Research questions

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- What are the roles of syntactic and semantic knowledge in proof production?
- What is the interplay between syntactic and semantic knowledge in proof production?
- Does the interplay between syntactic and semantic knowledge depends on the given proof and the mathematical concepts involved in the given proof?

In what follows sections in a smaller font are verbatim quotes from the interviews

Roles of Syntactic knowledge

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- Symbolic language helps defining concepts that are not particularly amenable to pictorial/geometrical representation.

  You see . . . no human can have a good intuitive geometrical or pictorial view of what the statement 'the sequence does not converge' means, for example. [. . .] Or say certainly no one can have a geometrical view of the statement 'this function is not uniformly continuous'.

Roles of Syntactic knowledge

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- Symbolic language is used as a tool for writing proofs and for manipulating formal statements:

  So for example the negation of a quantified statement I think is much easier as a symbolic definition. Because it is an algorithm: you replace 'for all' with 'there exists', you replace 'there exists' with 'for all' and then you get a statement and that is the algorithm.

Roles of Semantic knowledge

The mathematicians also talked extensively about **construction of meaning**

This is the definition and that is the meaning, and the meaning I construct is equivalent to the definition.

But how does this construction of meaning interacts with the formal definition of a mathematical concept?

Roles of Semantic knowledge

- Semantic knowledge is of great importance when there are part of proofs that require an act of choice from the part of the prover

It is the situation between the formal and the informal, I think. I mean . . . unless the student reaches ever the informal concept I think . . . to my mind it should be first very deeply ingrained in the student. And then it should be a justification in order to make sure that this is really doing what it ought to do this formal machinery.

Roles of Semantic knowledge

- Semantic knowledge grants deeper understanding of the mathematical concepts considered

Again ... for example that student of mine who said, you know, why ... why does it [*applying the formal definition*] prove convergence? The impression I get is that she would end doing it all, all the side calculations and everything, but she was approaching it because she knew this is what you are supposed to do to prove convergence, but she didn't really understand why she was doing it, I think.

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- Semantic knowledge grants flexibility in applying known concepts to new situations.

What is the interplay between syntactic and semantic knowledge in proof production?

Perhaps the most interesting finding emerging from the interviews is that mathematicians believe that **both syntactic and semantic knowledge are needed for successfully producing proofs and solving problems**.

They seem to point to a cyclic process based on drawing on syntactic and semantic knowledge in turn and often simultaneously. Syntactic knowledge is needed both to guarantee unambiguous use of the definition and to manipulate and produce a formal argument. In turn, semantic knowledge is needed to guide the syntactic proof production by drawing on insight into the main properties of the mathematical objects involved.

The role of the mathematical content

The data data however suggest also that the lecturers recognize that interplay between semantic and syntactic knowledge in proof production depends also on the required proof and the mathematical concepts involved.

We call *concept usage*

. . . the ways one operates with the concept in generating or using examples or doing proofs. (Moore, 1994, p 252).

The participants highlight at least four distinct types of concept usage.

The role of the mathematical content

- Concepts without initial pictorial representation for which resorting to syntactic knowledge is the only suitable approach.

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- Concepts for which syntactic knowledge is an effective tool.

  [*on using the quantified statement for convergence*] You can view this as the recipe, you can do this, you do this and you do this . . . You just follow the steps. And in some ways they are safer because . . . they will not make mistakes as long as they are technically doing the right steps.

The role of the mathematical content

- Concepts for which syntactic knowledge can be used for proof production but only ineffectively

  So the classic one in analysis is to say [. . .] 'there is N such that for all n bigger than N this implies that . . . ', let's say. And how do you exactly negate that? Well, if you negate it formulaically you end up with the kind of statement that is correct but it is not useful and it is not what you are going to use. You have to think: what does this mean now? And then write it down in words.

The role of the mathematical content

- Concepts for which syntactic knowledge alone cannot be used

  What struck me in marking these, and tutoring people doing this problem [*given a matrix* A*, find* $adj(adj(A))$], is that there is . . . this is really pretty hard . . . There are two completely different things you need. One is you need to be confident in these very formal manipulations of expressions and then at the right moment you need to use some common sense and say: 'Oh, the determinant of the diagonal matrix is obviously this' as a calculation and not as a formal manipulation that can be visualized. It is obviously this . . . this requires real thought.

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They perceive a spectrum which spans from concepts that cannot be used effectively by resorting to semantic knowledge only to concepts that cannot be used effectively by resorting exclusively to syntactic knowledge.

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The mathematicians also perceive (based on their experience of teachers, learners and 'do-ers' of mathematics) that both syntactic and semantic knowledge are necessary for successful proof production.