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ORIGINAL ARTICLE

Visualisation and proof: a brief survey of philosophical perspectives

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Abstract The contribution of visualisation to mathematics and to mathematics education raises a number of questions of an epistemological nature. This paper is a brief survey of the ways in which visualisation is discussed in the literature on the philosophy of mathematics. The survey is not exhaustive, but pays special attention to the ways in which visualisation is thought to be useful to some aspects of mathematical proof, in particular the ones connected with explanation and justification.

1 Foreword

It is a great honour to be asked to contribute to this special issue in memory of Hans Georg Steiner. He was a friend and mentor who had an enormous impact on the field of mathematics education, as acknowledged in the volume dedicated to him, *Didactics of mathematics as a scientific discipline*, which appeared in 1994 to mark both his 65th birthday and 20 years of work at the Institut für Didaktik der Mathematik (IDM) in Bielefeld.

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I had the great pleasure and privilege of meeting Hans Georg Steiner on several occasions, first in Bielefeld in 1983 and then at international conferences on mathematics education. I will always be grateful to him, in particular, for the opportunity to further my initial research on proof (Hanna, 1983) during two study periods I spent at the IDM in Bielefeld, both times at his invitation and with the support of the Deutscher Akademischer Austauschdienst (DAAD).

2 Introduction

A number of mathematicians and logicians have investigated the use of visual representations, in particular their potential contribution to mathematical proofs (Brown, 1999; Davis, 1993; Giaquinto, 1992, 2005; Mancosu, 2005). Over the past 20 years or so these investigations have gained in scope and status, in part because computers have increased the possibilities of visualisation so greatly. Such studies have been pursued at many places, such as the Visual Inference Laboratory at Indiana University and the Centre for Experimental and Constructive Mathematics (CECM) at Simon Fraser University in British Columbia. At most of these institutions, the departments of philosophy, mathematics, computer science and cognitive science cooperate in research projects devoted to developing computational and visual tools to facilitate reasoning.

A key question raised by the intensified study of visualisation is whether, or to what extent, visual representations can be used, not only as evidence or inspiration for a mathematical statement, but also in its justification. Diagrams and other visual representations have long been welcomed as heuristic accompaniments to proof, where they not only facilitate the understanding of a theorem and its proof, but can often inspire the theorem to be proved and point out approaches to the construction of the proof itself. And of course every mathematics educator knows that they are essential tools in the mathematics curriculum.

It is only in the last two decades or so, however, that visual representations have begun to be considered seriously as substitutes for traditional proof. Today there is still much controversy on the role of visual representation in proof, and a number of researchers are actively pursuing the topic. In their positions on this issue these researchers span a broad range. At one extreme are those who say that visual representations can never be more than useful adjuncts to proof, as part of their traditional role as facilitators of mathematical understanding in general. At the other extreme are those who claim that some visual representations can constitute proofs in and of themselves, rendering any further traditional proof unnecessary.

Between these two extremes one finds a variety of positions that are more nuanced or perhaps simply less clear. Some authors, for example, do not envisage a visual representation constituting an entire proof, but would maintain that an appropriate visual representation is acceptable as an integral component on which the proof as a whole would stand or fall. Other authors seem to be hesitant or inconsistent in their positions, and to this extent the division of the rest of this paper into three seemingly well-defined sections is of necessity somewhat forced. This paper is a brief survey of the ways in which visualisation and proof are discussed in the literature on the philosophy of mathematics; it does not look at the literature on visualisation and proof in mathematics education.

3 Visual representations as adjuncts to proofs

Francis (1996), for example, maintains that the increased use of computer graphics in mathematical research does not obviate the need for rigour in verifying knowledge acquired through visualisation. He does recognize that "the computer-dominated information revolution will ultimately move mathematics away from the sterile formalism characteristic of the Bourbaki decades, and which still dominates academic mathematics." But he adds that it would be absurd to expect computer experimentation to "replace the rigour that mathematics has achieved for its methodology over the past two centuries". For Francis, then, visual reasoning is clearly not on a par with sentential reasoning.

Other researchers have come to similar conclusions. Palais (1999), for example, is a mathematician at Brandeis University who worked on a mathematical visualisation program called 3D-Filmstrip for several years. Reporting on his use of computers to model mathematical objects and processes, he observes that visualisation through computer graphics makes it possible not only to transform data, alter images and manipulate objects, but also to examine features of mathematical objects that were otherwise inaccessible. Palais concludes that visualisation can directly show the way to a rigorous proof, but stops well short of saying that visual representations can be accepted as legitimate proofs in themselves.

4 Visual representations as an integral part of proof

Very few assert that proofs can consist of visual representations alone, but a number of researchers do claim that figures and other visual representations can play an essential, though restricted, role in proofs. Casselman (2000), for example, having explored the use and misuse of pictures in mathematical exposition, concludes that a picture can indeed form an essential component of a proof.

Taking his cues from Tufte (1983, 1990, 1997), Casselman first formulates eight suggestions for creating illustrations that foster mathematical understanding. Two of these are that "... the figures themselves should tell a story" and that one should "... ask constantly whether the figures really convey the point they are meant to" (Casselman, 2000, p. 1259). He goes on to give several examples of good mathematical illustrations, most of which adhere to the principles of visual explanation set out by Tufte (1997). The most striking is his comparison of the traditional picture for proving the Pythagoras theorem (see Fig. 1) with 16 pictures taken from a computer-animated series based upon the area-preserving property of shears. (This series is now familiar to many students.) He considers the animation to be measurably better than the traditional figure, which lacks explanatory power.

Casselman (2000) not only points out the importance of pictures for understanding, however, but goes on to state that "In spite of disclaimers and for better or worse, pictures—even if only internalized ones—often play a crucial role in logical demonstration" (p. 1257) and can "convey information, sometimes a whole proof " (p. 1260).

The term visualisation is most often applied to public acts of communication: using a diagram or other representation as a vehicle to convey a mathematical

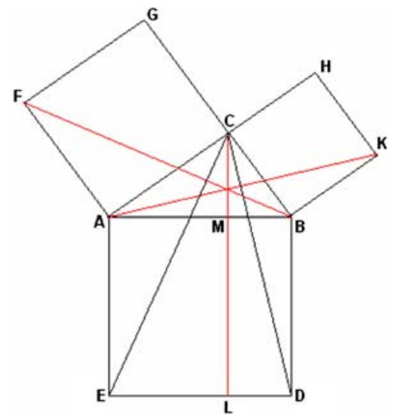

Fig. 1 Euclid's proof of the Pythagorean theorem

idea, to explain or convince. For Giaquinto (1992, 1993), however, visualisation is an individual experience that takes place in an internal mental space. Giaquinto, a philosopher of mathematics, is concerned with the epistemic aspects of this inner experience.

One of Giaquinto's prime interests is the use of visual imagination to discover mathematical truths. "One discovers a truth by coming to believe it in an epistemically acceptable way." (Giaquinto, 1992, p. 382) This conception of discovery allows Giaquinto to make two important claims. The first is that discovery requires independence: One must come to believe on one's own terms; one cannot blindly accept another's assertion. The second is that epistemic acceptability hinges upon a larger congruency: A discovery is not valid if it conflicts with other independently acquired beliefs.

Giaquinto differentiates between discovery and demonstration. One can believe in a discovery without having a valid justification (Giaquinto, 1992, p. 383). Alternatively, one might read a justification of a claim without being able to discover its truth. Thus he is interested in studying the role of visualisation in discovery without making any claim that it has a role in the construction of proof. He cites a few examples, among them the famous construction of the doubling of the square in the slave boy episode of Plato's *Meno* (see Fig. 2), to show that "the visual way of reaching the theorem illustrates the possibility of discovery without proof" (Giaquinto, 2005, p. 77).

Borwein and Jörgenson of CECM examined the role of visualisation in reasoning in general and in mathematics in particular, posing to themselves the two questions: "Can it contribute directly to the body of mathematical knowledge?" and "Can an image act as a form of 'visual proof'?" They answer both these questions in the positive, though they would insist that a visual representation can be accepted as a proof only if it meets certain criteria.

In arguing their position, Borwein and Jörgenson (2001) cite the many differences between the visual and the logical modes of presentation. Whereas a mathematical proof has traditionally been presented as a sequence of valid inferences, a visual representation purporting to constitute a "visual proof" would be presented as a static picture. They point out that such a picture may well contain the same information as the traditional sequential presentation, but would not show an explicit path through that information and thus would leave "the viewer to establish what is important (and what is not) and in what order the dependencies should be assessed." For this reason these researchers believe that successful visual proofs are rare, and tend to be limited in their scope and generalizability. They nevertheless concede that a number of compelling visual proofs do exist, such as those published in the book *Proofs without words* (Nelsen, 1993). As one example, they present the heuristic diagram, which aims to prove that the sum of the infinite series $1/4 + 1/16 + 1/64 + \cdots = 1/3$ (see Fig. 3).

Borwein and Jörgenson suggest three necessary (but not sufficient) conditions for an acceptable visual proof:

- Reliability: That the underlying means of arriving at the proof are reliable and that the result is unvarying with each inspection.
- Consistency: That the means and end of the proof are consistent with other known facts, beliefs and proofs.
- Repeatability: That the proof may be confirmed by or demonstrated to others.

One might wonder whether these criteria would not apply to proofs in general, not only to visual ones. One might also object that the first criterion in particular,

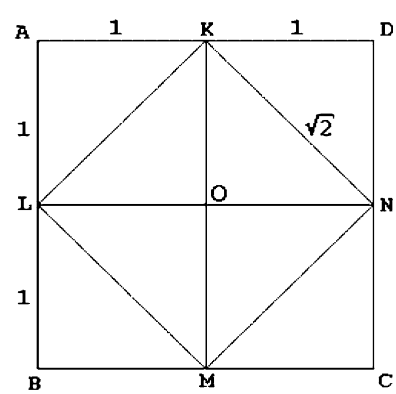

Fig. 2 Doubling the square

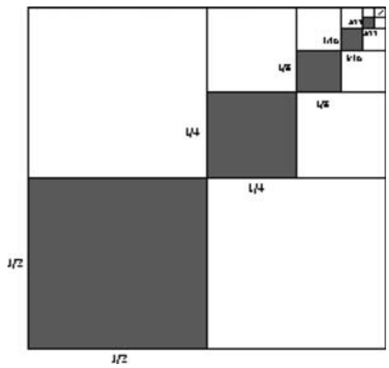

Fig. 3 Visualising an infinite series

lacking as it does a definition of ‘‘reliable,’’ does not provide enough guidance in separating acceptable from unacceptable visual proofs. Indeed, Borwein and Jörgenson make no claim to have answered this question definitively. Nevertheless, they would not only assign to visual reasoning a greater role in mathematics in general, but would also maintain that some visual representations can constitute proofs.

Davis (1993) advocates a conception of theorem that includes images, figures and computer graphics. He offers a ‘‘wide definition’’ of visual theorems that includes (a) all the results of elementary geometry that appear to be intuitively obvious, (b) all the theorems of calculus that have an intuitive geometric or visual basis, (c) all graphical displays from which pure or applied mathematical conclusions can be derived almost by inspection and (d) graphical results of computer programs which the brain organizes coherently in a certain way (Davis, 1993, p. 336).

By ‘‘theorem’’ Davis seems to mean the statement of a mathematical result, not of a justifying argument. Nevertheless, his discussion indicates that he often sees a figure as being explanatory enough to make another proof redundant. For example, he claims that the theorem that circles cannot be plane-filling inside a circle is made trivial by a figure of a circle containing three smaller free-floating circles (Davis 1993, p. 337–338). In the case of fractals, Davis states that the ‘‘visual theorem’’ gives us information about the mathematical objects that may be difficult or impossible to put into words. He says that the figure is ‘‘the passage from the mathematical iteration to the perceived figure grasped and intuited in all its stateable and unstateable visual complexities’’ (Davis, 1993, p. 339).

Although Davis does not dwell on this point, in both these cases he would appear to believe that the figure must be accompanied by a verbal or formulaic presentation. A circle containing three circles, or a fractal image, no matter how visually evocative, would not constitute a piece of mathematics. Davis (1993) is perhaps best read as asserting that a figure, because of its explanatory value but in addition to it, could be an integral part of proof.

5 Visual representations as proofs

Other researchers go further when challenging the idea that visual representations are no more than heuristic tools. Barwise and Etchemendy (1991, 1996) sought ways to formalize diagrammatic reasoning and make it no less precise than deductive reasoning. They acknowledge that the notion of proof as a derivation, consisting of a sequence of steps leading from premises to conclusion by way of valid reasoning, and in particular the elaboration of this notion in mathematical logic, have contributed enormously to progress in mathematics. They claim, however, that the focus on this notion has led to the neglect of other forms of mathematical thinking, such as diagrams, charts, nets, maps, and pictures, that do not fit the traditional inferential model. They also argue that it is possible to build logically sound and even rigorous arguments upon such visual representations.

These two researchers proceeded from what they call an informational perspective, building upon the insight that inference is ‘‘the task of extracting information implicit in some explicitly presented information’’ (Barwise and Etchemendy, 1996, p. 180). This view leads them to a criterion for the validity of a proof in the most general sense: ‘‘As long as the purported proof really does clearly demonstrate that the information represented by the conclusion is implicit in the information represented by the premises, the purported proof is valid’’ (p. 180).

The authors go on to say that whenever ‘‘there is structure, there is information’’, and that a visual representation, which may be highly structured, can carry a wealth of information very efficiently. Because information may be presented in both linguistic and non-linguistic ways, they conclude that strict adherence to inference through sentential logic is too restrictive, inasmuch as sentential logic is only a linguistic representation.

The question is how to extract the information implicit in a visual representation in such a manner as to yield a valid proof. Barwise and Etchemendy show examples of informal derivations, such as the use of Venn diagrams, and suggest that perfectly valid visual proofs can be built in a similar fashion upon the direct manipulation of visual objects. Unfortunately, as they point out, the focus on sentential derivation in modern mathematics has meant that little work has been done on the development of protocols for derivation using visual objects, so that there is much catching up to do if visual proof is to realize its considerable potential.

Though the view of these researchers is that proof does not require sentential reasoning, they do not believe that visual and sentential reasoning are mutually exclusive. On the contrary, much of their work has been aimed at elaborating the concept of ''heterogeneous proof.'' Indeed, Barwise and Etchemendy (1991) have developed *Hyperproof*, an interactive program which facilitates reasoning with visual objects. It is designed to direct the attention of students to the content of a proof, rather than to the syntactic structure of sentences, and teaches logical reasoning and proof construction by manipulating both visual and sentential information in an integrated manner. With this program, proof goes well beyond simple inspection of a diagram. A proof proceeds on the basis of explicit rules of derivation that, taken as a whole, apply to both sentential and visual information.

Few philosophers of mathematics make the explicit claim that diagrams or other visual representations can constitute an independent method of justification. One of the strongest advocates of this position is Brown (1997, 1999), whose stance is closely related to his Platonist view that mathematics deals with real objects having an independent existence. For him ''*Some 'pictures' are not really pictures but rather are windows to Plato's heaven*'' (Brown, 1999, p. 39).

In this context, Brown presents a number of theorems concerning sums and limits, and follows the statement of these theorems with ''picture-proofs''. Each of these consists of a single figure. He then gives a traditional proof for comparison. Brown's presentation implies that he believes these figures alone constitute proofs on the same level as the traditional arguments that follow them, consistent with his stated position (Brown, 1997, p. 169–172; Brown, 1999, p. 34–7).

It is not entirely clear how Brown comes to his conclusion that some visual representations constitute proofs. He seems to be using an analogy: just as proofs can be convincing and explanatory, so too figures can be convincing and explanatory. In essence, Brown appears to believe that a proof is anything that is both convincing and explanatory – and thus that any visual representation which satisfies those two criteria is a proof.

Just by looking at them, however, one cannot even understand how most of these ''picture-proofs'' function as representations of mathematical objects, much less as valid mathematical arguments. Generally one finds that one has to apply to them a reasoning process, in the form of sentences, in order to understand the theorem and be convinced of its validity. Brown fails to make the case that this reasoning process, a traditional mode of mathematical thinking, is unnecessary.

Folina (1999) provides useful and succinct criticisms of Brown's account, concluding that ''... not every kind of convincing evidence for a mathematical claim counts as a proof. In particular, Brown does not show that a picture, or anything 'picture-like', can be a proof. In my view, he does not really argue for this'' (p. 429.)

6 Summary and conclusions

This brief survey shows that we are still far from a consensus on all potential roles of visualisation in mathematics and mathematics education, and in particular on its role in proof. Its role as an important aid to mathematical understanding is universally accepted, however, and there is room for more effort aimed at better ways to use visualisation in this role.

One can legitimately ask, in particular, in what ways visualisation can be most useful to the aspects of mathematical proof important to mathematics education, particularly those connected with explanation and justification.

As stated in the introduction, this paper set out to survey recent work on visualisation and proof in the literature on the philosophy of mathematics, not in the literature on mathematics education. In the latter as well, however, there is a large body of comment on the important role of visualisation in mathematics education. As examples one might cite the works summarized in Presmeg (2006) and in particular the work of Tall (2002), along with his papers of the last 20 years (listed in http://www.davidtall.com/).

There are many research papers in mathematics education that discuss in particular the role of both diagrammatic reasoning and dynamic geometry in the learning and teaching of proof—see, for example, the special issue of *Educational Studies in Mathematics* on proof in dynamic geometry environments edited by Jones, Gutiérrez and Mariotti (2000). The papers in this issue showed, through classroom research, that dynamic software was very successful in enhancing the ability of students to notice details, to conjecture, to reflect on and interpret relationships and to offer tentative explanations and proofs. The literature in mathematics education lends support to the view that emerges from the literature in mathematics itself, that visualisation is a very important aid to mathematical understanding.

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