CONCEPT USAGE IN PROOF PRODUCTION: MATHEMATICIANS' PERSPECTIVES

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In this paper I draw on mathematicians' perspectives on students' concept usage in proof production. Their analysis points to a spectrum of concept usages in proof: from concepts which are more effectively used syntactically as they lack an immediate pictorial representation to concepts which are more effectively used semantically as their syntactic representation is too complex to use. This suggests that, together with knowledge of mathematical objects and proof techniques, students need to acquire a meta-skill: to be able to recognize which is the most effective way to precede in proof and how this choice depends also on the mathematical concepts involved in the given proof.

INTRODUCTION

The contribution of this paper to the ICMI19 Study is situated at tertiary level and aims at gaining insight into mathematicians' perceptions of their students' efforts in producing proofs via an analysis of the students' written work. Their insights (as teachers and as researchers in mathematics) are valuable in the sense that they offer an analysis of what skills the mathematicians believe students should have in order to be successful in producing proof. In this paper I will focus on one of the skills that mathematicians view as important, namely recognizing when to proceed *semantically* and when to proceed *syntactically* (Weber 2001, Weber and Alcock, 2004), and how such choice depends also on the specific mathematical objects that appear in a given proof.

I will focus on *concept usage*

> ... which refers to the ways one operates with the concept in generating or using examples or doing proofs. (Moore, 1994, p 252).

In what follows I will describe mathematicians' insights on concept usage in proof production.

THE STUDY

The data I present originate from a study[1] with mathematicians (indicated in the interview extracts with capital letters) from across the UK as educational co-researchers. The research engaged university lecturers[2] of mathematics (more details on the participants to the study can be found in Iannone & Nardi, 2005) in a series of Focused Group Interviews (Wilson, 1997), each focusing on a theme

[1] Supported by the Learning and Teaching Support Network in the UK, with Elena Nardi.

[2] In the text we refer to the participants of the study as Lecturers. Meanings of this term differ across different countries. We use it here to denote somebody who is a member of staff in a university mathematics department involved in both teaching and research.

regarding the teaching and learning of mathematics at university level that the literature acknowledges as seminal. Two of the themes revolved around proof production. For each interview a Dataset was produced. This included a short literature review and bibliography, samples of student data (e.g. students' written work, interview transcripts, observation protocols) and a short list of issues to consider. The analysis of the interview transcripts largely followed Data Grounded Theory techniques (Glaser and Strauss, 1967) and resulted in thematically arranged sets of Episodes – see elsewhere (e.g. Iannone & Nardi, 2005) for more details.

THE DATA

In the interviews the participants discussed extensively concept usage in proof production. Their views, taken as *expert views*, contribute to the understanding of which skills students need to acquire early on in undergraduate mathematics. The data I present below suggest that the lecturers recognize that different mathematical concepts call for different usages in proof production. Furthermore, the interplay between semantic and syntactic concept usage depends also on the required proof. The participants highlight at least four distinct types of concept usage.

Concepts without initial pictorial representation for which resorting to syntactic knowledge is the only suitable approach

These are concepts which are very difficult to represent via mental images, for example via something that can be drawn on a number line or a Cartesian plane. These concepts can be used in proof production only via manipulation of the syntactic statement that defines them. One example is given below

> E: A classic example that arises in analysis all the time is "N arbitrarily large". [...] And it is a very sophisticated notion, the idea that all the quantities I am talking about are finite, but they are arbitrarily large [...] one clearly needs to express [*this*] symbolically. The property that you might ascribe to this arbitrarily large number, the modulus of which is less than that and so this happens ... in symbols and there is no way around it. Or at least, I am not aware of a way around it.

Concepts for which syntactic knowledge is an effective tool

These are concepts which can be used semantically, but while it is possible to have a visual representation, a syntactic approach is more effective. An example is the use of the negation of the statement "the sequence converges" (as discusses in the context of Example 1 in Appendix).

> A: ... working with intuitively geometric pictures, when it comes to convergence, is something that is very private: some people work like this and some don't. And I can well imagine that there are students that can work along a string of quantifiers. They just do what they are told. You can view this as the recipe, you can do this, you do this and you do this... You just follow the steps. And in some ways they are safer because ... they will not make mistakes as long as they are technically doing the right

steps. So, you see, this depends on how you can think about mathematics and we all are different in how we can see this things and how we can work with it.

E: Sort of... I push very, very hard number lines, that they [*the students*] should draw number lines, have pictures and so on... but... You see, no human can have a ... good intuitive geometrical or pictorial view of what the statement "the series does not converge" means, for example. I don't think... Or say certainly no one can have a geometrical view of the statement "this function is not uniformly continuous", let's say.

This extract suggests a spectrum of concept usages: from those that are used effectively semantically to those that are used effectively syntactically in proof production. Furthermore, in resonance with the literature (see for example Burton (2004)), the exchange above suggests that concept usage not only depends on proof and concept but also on personal proving preferences.

Concepts for which syntactic knowledge can be used for proof production but only ineffectively

These are concepts for which one could resort to syntactic manipulation, but this approach is not helpful in practice. This process leads to statements that are syntactically correct, but hard to work with. An example is offered by Lecturer E:

E: They [*the students*] have to get used to it. To negate the statement [*of convergence of a series*] ... you can algorithmically negate changing "there exist" with "for all" and so on ... And then you come up with a very weird statement, not the sort of statement you can actually reason with effectively. You need to modify it by resource to its actual meaning. So the classic one in analysis is to say [...] "there is N such that for all n bigger than N this implies that ...", let's say. And how do you exactly negate that? Well, if you negate it formulaically you end up with the kind of statement that is correct but it is not useful and it is not what you are going to use. You have to think: what does this mean now? And then write it down in words.

In such situations semantic knowledge needs to come into play at the moment when syntactic knowledge has failed to produce meaningful statements.

Concepts for which syntactic knowledge alone cannot be used

These are concepts for which the use of syntactic knowledge by itself will lead towards higher symbolic complexity and away from the production of the proof requested. An example is given in the extract below, in the context of Exercise 2 (see Appendix):

E: What struck me in marking these, and tutoring people doing this problem [*given a matrix A, find adj(adj(A))*], is that there is ...this is really pretty hard ... There are two completely different things you need. One is you need to be confident in these very formal manipulations of expressions and then at the right moment you need to use some common sense and say: "Oh, the determinant of the diagonal matrix is obviously this" as a calculation and not as a formal manipulation that can be visualized. It is obviously this ... this requires real thought.

A: In fact, it sort of does the thing which … it is almost like suspending the definition that is required before you can do anything. If you go away and work out the adjugate of the adjugate you are dead … you are really stuck.

E: There are not enough sigmas in the world!

(all participants laugh)

A: You will never understand this. So you must suspend things and start to rely on algebraic manipulative insight and then you realize that after all you can do something.

This situation is summarised by Lecturer C (see also Moore, 1994, p 258 on the need to use definitions to guide through proofs):

C: … which is ironical because just before we were talking about how students cannot start from the definitions and you just realise that they don't have to start from the definition for this homework [*Exercise 2*]. At some level students have to start from the definition and then they have to understand when not to start from the definition.

[..]

A: It is almost if… if you are successful at this question you… you have almost acquired a meta-mathematical ability, namely you are required to look out for the things that would be helpful rather than trying to understand what is this blooming thing and what is this and so on. So you are looking for the strategy by which you can enter and apparently [*win*] this battle.

In the extract above Lecturer A points towards the acquisition of what he calls a '*meta-mathematical ability*' namely the ability to recognize which of the ways to proceed (syntactically or semantically) is best suited to the problem at hand, and that prescriptions for tackling proof are not effective.

DISCUSSION

Several interesting points emerge from the data above. The participants are aware of the existence of semantic and syntactic aspects of concept usage for proof production. This is perceived as a spectrum which covers concepts that cannot be used effectively by resorting to semantic knowledge only to concepts that cannot be used effectively by resorting exclusively to syntactic knowledge. The reasons put forward for the existence of this spectrum include the absence of pictorial representation for some concepts and the complexity of syntactic representation for others. The *meta-skill* that undergraduate students need to acquire is to recognize the effective way to operate with a mathematical concept in a given proof (see also Weber, 2001). Of course, as we have seen in some of the extracts, the personal preference of the prover plays a big part in this process while various proof strategies for undergraduate mathematics are well established. There is agreement among the participants, as observed also in Alcock and Weber (2005), that is possible to be successful in elementary analysis by resorting just to syntactic knowledge. At the same time the participants suggest that this strategy cannot be sustained, and that there are proofs, even in first year mathematics, that cannot be completed by proceeding purely syntactically.

Another interesting point emerging from the data is that the participants do not distinguish between skills required for proof production at undergraduate level and those at more advanced level. Their observations originate from their own experience of research mathematicians. In the first interview extract Lecturer E, an analyst with several years of experience as a researcher, talks about a concept which has only syntactic existence for him even now, not just for students at the beginning of their studies. This observation corroborates the idea that to study expert behavior in doing mathematics, such as the behavior of a research mathematician, can help understanding what skills students need to acquire to become experts themselves.

CONCLUDING REMARKS

From the data presented in this paper it emerges that mathematicians ascribe defined roles to syntactic and semantic knowledge in proof production. However, on the basis of their experience as learners, researchers and teachers, they argue that in order to produce proofs successfully, learners need to be able to move along this spectrum between the two modes in order to adapt their proof production to the type of mathematical problem in question and to the mathematical concepts involved in the proof. Hence, for a student to become proficient in proof production, the skill to tailor their proof behavior to the type of mathematical problem in question becomes indispensable. Moreover, each learner (be it a student or a mathematician producing new mathematics) may have a favorite learning style, but needs to recognize that the preference for one type of action over the other must come with the realization that it is necessary at times to adapt and leave the preferred mode of action behind in favor of a more effective one.

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APPENDIX

Exercise 1: Write out carefully the meaning of the statement "the sequence $\{a_n\}$ converges to A as $n \to \infty$."

Exercise 2: Suppose $n \geq 2$ and A is an $n \times n$ matrix with $\det(A) \neq 0$. In the following adj(A) denotes the *adjoint* (or adjugate) matrix of A.

1. Use the fact that $(\mathrm{adj}(A))A=\det(A)I_n$ and the product formula for determinants to show that $\det(\mathrm{adj}(A))=(\det(A))^{(n-1)}$.
2. Prove that $\mathrm{adj}(\mathrm{adj}(A))=(\det(A))^{(n-2)}A$.