

Taken-as-shared: a review of common assumptions about mathematical tasks in teacher education

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Taken as shared

The call for outline papers to be considered for inclusion in this special issue elicited 111 offers. This presented a formidable problem for the three editors, ourselves and Orit Zaslavsky who contributes the final paper in this collection. During our reading of the submissions we identified many aspects of working on mathematical tasks with teachers which seem to be common and which were frequently referred to in the literature.

In the interests of offering a special issue which gives an up-to-date description of the field, and which moves forward from currently shared practices, we summarise these aspects here in an introductory paper rather than include them in individual papers as if they are somehow novel. In this paper we shall describe elements of the design and use of mathematics-related tasks¹ with teachers, whether in a pre-service or in-service structure, which appear to be *taken as shared*. We do not claim that everyone conforms with, or agrees with everything that is here, but our summary spans the submissions received and the wide range of perspectives taken by the authors. Even where authors use different theoretical perspectives to think about learning and teaching, there is considerable agreement about what educators hope will be achieved through task use and about the methods of use. These elements pervade the literature, and we provide references in this paper to places where elaboration can be found.

In the second part of this paper we give an overview of the range of practices and issues raised in the whole set of papers submitted. To design a suitable set of papers from the outlines submitted, we invited some authors to write longer papers which, in our view, present significant theoretical contributions to the field, together with implications for

¹ Throughout, *task* means mathematics-related task.

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practice, and other authors to present shorter papers which further supplement or elaborate current thinking. In the final paper, Orit Zaslavsky conceptualises this final collection.

Taken-as-shared assumptions

This special issue of JMTE is focused on the use of mathematics-related tasks with teachers, how that might influence their subsequent pedagogy and how educators develop their own expertise in the design of mathematics-related tasks for teachers. It inevitably links the learning of mathematics in the classroom to teachers' awareness both of themselves as learners and of the ways in which learners differ in how they learn mathematics.

Task and activity

Hiebert and Wearne (1993) proposed that 'what students learn is largely defined by the tasks they are given' (p. 395) and we take this to apply to pre-service and in-service teachers as well. We follow Christiansen and Walther (1986) in distinguishing between mathematics-related tasks and the activity which arises as a result of engaging with tasks. As Christiansen and Walther frame it,

... the problem is to identify means by which the teacher may promote a unified conception – within the learner – of the role of task-and-activity, of learning, of mathematics, and of his personal, conscious control of his own learning process. (ibid, p. 264)

We include as 'activity' what the learner actually does, interaction with other learners, interaction with other resources and interaction with the teacher. It is often relevant to distinguish between the task as conceived by the author, as interpreted and intended by the teacher (if she is not the author), and as interpreted and constructed by the learner.

... even when students work on assigned tasks supported by carefully established educational contexts and by corresponding teacher-actions, learning as intended does not follow automatically from their activity on the tasks. (ibid, p. 262)

Comenius' view that '[learners] should get accustomed to penetrating to the real roots of things and take into [them]self their true meaning and usage, rather than read, perceive, memorize, and relate other people's opinions' [quoted in Ulrich 1947, p. 344] was reformulated by Freudenthal (1973, p. 110) as '[t]he best way to learn an activity is to perform it'. This applies as much to pre-service and in-service teachers as to young children. The way a mathematical task is presented, developed, worked with and drawn to a conclusion instantiates the teacher's mathematical *weltanschauung*, including beliefs about and attitudes towards mathematics, its learning and how it can be taught (Thompson 1992). This has a strong influence on, but not a direct causal connection with, the sense that learners make (Stein et al. 1996, p. 459). Any task lies on a spectrum between an open invitation and direct instruction. What matters is not so much the task itself, or the applicability of the task, but 'the power to evoke a mathematical response from the [learner]' (Fletcher 1964, p. 1).

The task as specified or undertaken is the explicit or outer task. The implicit intention of the task, that is, what it affords of mathematical themes, concepts, theorems, connections to other topics and techniques, multiplicity of approaches, interpretations and

re-presentations, has been referred to as the *inner task* (Tahta 1980). Making the inner task explicit converts it into an outer task and runs the risk of directing attention so that students do not actually experience the purpose for themselves. In addition to the inner and outer tasks there is a *meta task* consisting of opportunities to work on personal dispositions and propensities such as doing whatever first comes to mind, waiting to be told what to do, or diving into the first approach that comes to mind (Dreyfus and Eisenberg 1991; Mason 1992; see also Mason and Johnston-Wilder 2004b, pp. 241–242).

Factors which influence the effectiveness of a task in promoting the intended kind of activity include ethos and atmosphere; established practices and ways of working; students' expectations of themselves and of each other as influenced by the system and their pasts; and learners' sense of self-confidence, agency (mathematically and socially) and identity. Following Brousseau (1997) we can refer to these as the *milieu*. While recognising that other traditions might have slightly different descriptions it is common to all that tasks do not in themselves generate learning, but are initiating, structuring or framing devices for pedagogy and learning. The notion of milieu is particularly useful as it includes the intellectual affordances and constraints of the task.

Searching for definitive influences in mathematics teacher education has led to interest in many different aspects of the milieu, the situation, the classroom context, as well as the structure of the tasks themselves. In focusing on mathematical tasks we retain this complex view of teaching sessions and activity, but do not include papers which foregrounded aspects other than tasks, such as papers which described course structures, or the norms of taught sessions.

Task in the full sense includes the activity which results from learners embarking on a task, including how they alter the task in order to make sense of it, the ways in which the teacher directs and redirects learner attention to aspects arising, and how learners are encouraged to reflect or otherwise learn from the experience of engaging in the activity initiated by the task. Reflection, discussion, individual and group work, time to ponder and the use of resources such as ICT or other apparatus are integral to the ways of working on tasks.

Ways of working in teacher education

In reading the 111 submitted proposals, we learned that teacher education programmes and projects throughout the world use similar ways of working with both pre-service and in-service teachers. We can assume, therefore, that research-informed teacher education programmes and projects throughout the world include prompting teachers to

- engage in mathematical thinking through working on mathematics-related tasks, with the aim of bringing it to teachers' attention that effective mathematical tasks promote and release mathematical thinking and can afford opportunities for bringing aspects of mathematics and mathematical thinking to the attention of learners;
- reflect on the experience of doing mathematics-related tasks oneself, or cooperatively, including becoming aware of multiple approaches, perspectives and strategies with the aim of teachers' developing the habit of adapting mathematical tasks so as to enable them to listen to learners and to develop sensitivity to learners' thinking and obstacles to that thinking;
- develop and try out frameworks for both pedagogy (general teaching) and didactics (specific to mathematical topics), such as analysing task structure, including purpose

- and affordances (whether pedagogic, or exposure to mathematical themes, mathematical powers, mathematical concepts, etc.) with the aim of making principled choices of tasks and interaction strategies when working with learners;
- consider implications for teaching, including designing and trying out related mathematics-related tasks with learners, as well as extending and varying task structure and task presentation (teaching experiments) with the aim of developing the habit of ongoing innovation, observation and reflection;
 - observe and analyse teaching (own, videotape, other), text-based tasks, etc. with the aim of raising pedagogic and didactic issues, and prompting reflection;
 - experience opportunities to observe and listen to learners (live, during interactions with them, on video, etc.) with the aim of learning from listening;
 - challenge procedure-dominated approaches which depend on rote memorisation and mechanical use of routines and algorithms with the aim of challenging teachers to make principled choices based on their learners' needs;
 - challenge their (memories of) classroom experiences as students themselves as the sole possibility for their teaching, with the aim of liberating them to be innovative and experimental;
 - support them in appreciating and making use of theoretical constructs which sensitise people to useful distinctions and relationships, which in turn can inform practice through bringing to mind alternative choices with the aim of promoting principled choices of tasks and interaction strategies.

One of the tensions which arises in working with teachers is that teachers are often on the lookout for something they can use in their classrooms, perhaps immediately. Thus tasks offered to teachers for their personal development are sometimes interpreted either as tasks to take straight into the classroom or rejected as 'not relevant to my teaching'. Tasks are often designed so that teachers can experience for themselves at their own level something of what their learners might experience and hence become more sensitive to their learners. The fundamental issue in working with teachers is to resonate with their experience so that they can imagine themselves 'doing something' in their own situation, through having particularised a general strategy for themselves, rather than relying on being given particular 'things to do'.

Although effective mathematical tasks for teachers share many of the features of effective mathematical tasks for learners, tasks for teachers also serve a higher-order purpose. To become an effective and professional mathematics teacher requires development of sensitivities to learners through becoming aware of one's own awarenesses. For example, school topics can be seen as the result of formalising informal actions, through becoming explicitly aware of those actions (Gattegno 1987; Piaget 1972, pp. 37–38); becoming a teacher involves becoming aware of how this happens and how it can be brought to happen (Mason 1998). As Piaget remarked, 'intelligence organises the world by organising itself'. What is central to teacher education is that for teachers, learning and action are one and the same: their professional choices of actions are the manifestation of what they have learned or are learning.

What learners are doing

In thinking about the role of mathematical tasks we find common features, even among those coming from different perspectives. Individuals, including individual teachers, make

sense (literally) based on past experience and on what comes to mind in the moment, through participation in socio-cultural practices, influenced by the institutional environment, language and ethos of the school and of the particular classroom. Engaging learners in activity is important, but in order to learn from that activity they need to experience some kind of shift or transformation in what they are sensitised to notice and attend to mathematically; the scope and structure of their concept images (Tall and Vinner 1981) and example spaces (Michener 1978); their fluency and facility in using mathematical procedures, representations and formats; their awareness of links and connections between mathematical topics; their awareness of where, and in what kinds of situations, different ways of thinking might be relevant; and so on. Interaction with peers and with more experienced 'others', through engaging with tasks provided by experts, or through more direct dialogue, may be essential for making such shifts. It is certainly important in formulating, articulating and internalising generalities, abstractions and pedagogical principles (Vygotsky 1978). These are often achieved either through encountering new, strange, unexpected phenomena—described as *cognitive dissonance* in the Piagetian tradition (Bell 1986, 1987; Bell and Purdy 1986; Festinger 1957, p. 3). They can also be achieved through reflection (e.g. reflective abstraction Piaget 1980, pp. 89–97) particularly when guided and structured with others (Gattegno 1987, p. 40). Reflection as a learning habit can, like all habits, become a normal way of engaging with tasks.

What teachers do

In order to be an effective teacher it is necessary to step back from engagement in mathematics and from acts of teaching, and to become aware of learners as active construing agents who work both by and for themselves, individually and also in concert, defining and being defined by, positioning themselves and being positioned by their milieu. All the outlines we read were working with a considerably more complex model of teacher-knowledge than the commonly made distinction between *subject content knowledge* and *pedagogic content knowledge* (Shulman 1986), which is neither complete, clear nor definite (Hiebert et al. 2002). These notions need to be augmented by, among other things, understanding how being knowledgeable about mathematics teaching influences classroom actions and knowing to act in the moment through having pertinent possibilities come to mind.

Sources of these assumptions: the groundwork for theory

In this section, we draw on a wide range of perspectives and show how these relate to the mathematical tasks in mathematics teacher education. In doing so, we are focusing more on common features than differences, and how different traditions articulate these.

Jean Piaget (1970, 1971) and his critics remind us of the individual's drive to make sense of experiences, and how this develops and matures over time. We also learn that great care is required in interpreting learners' behaviour, for part of that behaviour may be in response to the sense they are making of our probes (Donaldson 1978), and part may be a display of the practices into which they have been enculturated and or which they think they are expected to display. This lesson carries over into working with adults as well: the question asked and the task proposed, as construed by the proposer, may not be what the learner construes. There may be a hidden or covert curriculum of unexpressed practices and norms which learners are expected to pick up (Snyder 1970) or that learners think they

are expected to display. Put another way, people's behaviour is principled, but sometimes those principles may be obscure, constructed as an amalgam of practices encountered previously (e.g. Brown and van Lehn 1980; van Lehn 1989). Learning to listen-to learners, rather than simply listening-for what we hope to hear (Davis 1996), is a core part of becoming and developing as a teacher.

Lev Vygotsky and subsequent colleagues remind us that, among other things, cultural tools mediate between individuals and ideas (Vygotsky 1978), that activity is the basis for generating experiences and that higher psychological processes are experienced through 'legitimate peripheral participation' (Lave and Wenger 1991) in activities involving more expert colleagues. Moreover, we are essentially actively learning beings, so educators need to structure situations in which teachers can make sense actively of the tasks of teaching. To do this, Hans Freudenthal (1973, 1983) and colleagues remind us that mathematics develops from reflection on, and questioning about, phenomena. Unless the phenomena of teaching and learning can become real for teachers, they are unlikely to make sense to them, just as mathematics has to become real for their students. The notion of 'what is real' for someone refers to their inner state and not necessarily to actions in the material world. An additional problem affecting all authors in this volume is that the institutionalisation of teacher education requires that the sessions described are not taking place within the relevant environment: the tasks and activity are not the tasks and activity of teaching; the teacher education milieu is not the classroom; 'reality' has to be about internalisation rather than immediate action.

Lee Shulman (1986, 1987) claimed that knowledge for teaching includes among other things, knowledge of mathematics, knowledge of general pedagogy, and *pedagogical content knowledge* specific to the teaching of mathematics generally, and individual topics particularly. We follow the European usage and refer to *mathematical pedagogy* as the collection of strategies and detailed ways of working with learners on mathematics across topics, and *mathematical didactics* as the collection of strategies, cultural pedagogic tools and associated psychology to do with learning particular mathematical topics at the level of individual concepts, techniques and properties. As indicated earlier, Shulman's distinctions are not necessarily useful for the task of educating mathematics teachers, and many of the papers in this collection present a more activity-based knowledge than he implied. We find more authors take a view similar to that of Davis and Simmt (2006) in maintaining a complex view of knowledge, identity and practice. Nevertheless, Shulman's ideas are still stated as the foundation of much practice in teacher education.

An approach to theorising learning relationships is provided by Guy Brousseau and colleagues (Brousseau 1984, 1986, 1997). There is always a didactic contract between teacher and learners, giving rise to an inevitable tension: the more clearly the teacher (or teacher educator) indicates the behaviour sought, the easier it is for learners to display that behaviour without generating it from themselves, that is, without learning. Following Yves Chevallard (1985), whenever a task is designed for someone else, there is a transposition in which the expert's awareness is transformed into instructions in behaviour for the learner. This highlights the importance of developing ways of working, a classroom rubric in which the learners are drawn into patterns of thinking, in which some transforming action takes place. Through a process of scaffolding and fading (Brown et al. 1989) learners initiate for themselves actions which were initiated in purposeful teaching. The added problem for teacher educators is that these actions take place somewhere else, in another context. Richard Skemp (1976) reminds us that understanding can be largely instrumental, largely relational, or some combination. This applies to learners of all ages, including pre-service and in-service teachers. The *transposition didactique* and the *contract didactique* are

amplified by a culture of local target setting to create a temptation to make the minimum investment needed to get through tasks and assessment. An added problem for teacher educators is that expecting teachers to develop relational knowledge of mathematics which has, for them, previously only been instrumental can be seen as ‘more stuff to do’ rather than underpinning procedures with meaning (Skemp 1976). The teacher educator may be tempted to transpose the complexities of teaching into injunctions about what to do; it is a feature of the papers in this collection that the complexities of teacher education are maintained. Important illustrations of the tendency to simplify are given by Mary Stein and colleagues (Stein et al. 1996; Stein et al. 1999). Tasks which are intended to generate several solution strategies, multiple representations and discussion may be modified during classroom interaction so that the cognitive demand is reduced as work progresses. In their study teachers resisted complexity and challenge, adapting so that tasks became quick and obvious (1996, p. 462). The extent to which high-level cognitive demand is intentionally structured and maintained by the teacher varies from country to country, and institution to institution (Hiebert et al. 2003).

The psychosomatic structure of the human being is seen as autopoietic (Maturana 1988) and this is assumed to carry on into adult life. So as a teacher you can train learners to perform techniques in stylised contexts, but you need to harness their natural powers (Mason and Johnston-Wilder 2004a), described as ‘spontaneous’ in the Vygotskian discourse, in order that they ‘educate their awareness’ so as to respond creatively in novel situations. Caleb Gattegno (1987) reminds us that although behaviour can be trained, it is only awareness that can be educated. It is through imposing relations on our perceptions that we encounter our ability to make choices, and it is through becoming aware of our likes and dislikes concerning mathematical tasks and activities that we detect ‘the mind at work creating works of the mind’—which is seen as ‘mathematics’ by Tahta and Brookes (1966, p. 8). They remind us that if educators provide only apparatus and tasks, learners can abdicate from mathematics, because they are unaware of how to make sense of the environment by imposing mathematical relations upon it. Somehow, tasks offered to teachers need to afford opportunity for working on mathematical relations and for imagining how they might enable others to develop similar awareness.

A major way to do this is found in the work of Orit Zaslavsky. A central part of teacher interaction with learners is the presentation and working through of (instructional) examples, whether of mathematical objects or of problems and their solutions (Zaslavsky 1995). Examples or example creation tasks which spark uncertainty or similar disturbance are more effective than ones which do not upset current assumptions (Zaslavsky 2005). It follows that novice teachers benefit from engaging in tasks about examples, as well as tasks of various types, different ways of presenting tasks, sustaining work on them and drawing them to a close with pedagogical effectiveness.

We see the various contributions outlined above as components of a potential theory of the nature and role of mathematical tasks in teacher education.

How tasks are used

As well as selecting a few papers for publication, we used the outlines as a remarkable data set about the state of this field of interest; we felt ourselves privileged to see it. We undertook a content analysis of the whole collection of papers and this strongly informed the final choice. From the initial reading, about half of the proposals offered mathematical tasks, as we requested. There were other papers of good research quality describing tasks

for teachers which would be generic rather than subject specific, such as those which drew their attention to the value of discussion, or which focused on teaching mathematics to potential teachers without connecting this to their future practice. The papers we shortlisted all used tasks in complex ways which combined personal, collaborative, reflective and pedagogic engagement and thus took account of the theoretical matters discussed above. For example, some described systematic lesson study based on a mathematical task, with cycles of teaching, collaborative reflective evaluation, adjustment and re-teaching. There were several variations on applying teaching-comparing-reflecting cycles, organised so that each teacher has experience of using or observing 'the same' task in several different teaching contexts, or of using or observing slightly different versions of tasks. Several papers described ways in which teachers can reconstruct their own knowledge with pedagogy in mind, particularly to develop instrumental knowledge from their procedural memories. Others focused on students' typical responses to tasks, so that teachers can extend their knowledge of what is possible by trying to understand what learners perceive. Thus three ways of working with tasks were well exemplified: using tasks to understand learners' mathematics; using tasks to develop teachers' own mathematical awareness, either re-thinking their views of mathematics or re-experiencing learning new maths; using tasks to think about how different ways of teaching offer different mathematical affordances. Many contributions showed creative ways in which teachers could learn new mathematics for themselves while also thinking about the pedagogic implications of their experience. A few papers used tasks to engage teachers in thinking critically about tasks, such as analysing textbook tasks, or being asked to invent, try out and evaluate tasks. Comparisons between slightly different tasks, or the effects of different pedagogical strategies, are a possible way to learn more about the role of tasks in teaching.

We were looking for papers which problematised teacher education, implicitly or explicitly, in terms of mathematical task design for teacher education, the relationships between tasks and pedagogy, purposes of mathematical tasks, and selection and use of mathematical tasks for teaching.² None of these issues was fully treated among the shortlisted proposals, and we made the final selection in order to represent contributions to all these issues. There was little said, however, about suitability of tasks for classrooms, for teachers' own mathematical knowledge, or for other aspects of teacher education whether pre-service or in-service. The prior experience and nature of knowledge differ for each of these cases. Many papers appeared to assume that the same tasks would suit all purposes, which may have been true, but the associated pedagogy could have then been problematised and its effect on the resulting activity discussed. Only a few papers addressed mathematical tasks which help teachers make in-the-moment decisions, yet to us it is these incidents which can make the difference between effective, responsive teaching and mechanical 'task delivery' models. We chose not to include papers on the mathematical knowledge 'needed' by teachers, since this is dealt with extensively elsewhere (e.g. Ball 2000; Brousseau 1984; Davis and Simmt 2006; Freudenthal 1973; Mason et al. 1988; Rowland et al. 1998).

Most papers came from locations where it is still the dominant practice to teach mathematics and pedagogy separately, the emphasis often being to patch up this fissure. Some came from other traditions, including the complex integration of knowledge, pedagogy and practice such as happens in UK and Australia and these are fully represented in the collection.

² The entire selection process of the final papers for this special issue was carried out jointly by the three guest editors, namely, by the two of us together with Orit Zaslavsky.

Several authors implied principles for task choice for an inquiry context, recognising that the task needs to have the potential to be a vehicle for teachers' exploration of mathematics, a trigger for awareness about the processes of inquiry and an arena for curriculum mathematics. We think these are good principles, and the field needs to develop in becoming more articulate about how certain well-used tasks demonstrate these propensities and how to design tasks based on these principles. We question whether tasks need to be structured in ways which require 'inquiry' or whether instead 'inquiry' is the mindset with which teachers, and ultimately their students, need to approach all tasks. It is often assumed that exclusive emphasis on experiencing new forms of mathematical task helps both new and experienced teachers to introduce, to sustain and to defend rationally, new forms of practice within more conservative school contexts. Possibly, inquiry as a form of engagement with traditional tasks in traditional settings would be a more effective way of changing teaching within such contexts.

Most teacher education courses cannot possibly deal in depth with every topic the teachers are going to teach. Choice of focus is therefore an important issue; do educators offer extensive tasks on key mathematical ideas, do they attempt to 'fill gaps' in teachers' knowledge or do they provide exemplars of task types which can be applied to a range of content? Some educators offer mathematics that would be new to the teachers and hence give them an experience of learning (a fine example of this is the use of mod 99 arithmetic developed by Milan Hejny (Simpson and Stehlikova 2006)). Immersion in a microworld to experience long-term development of understanding through complexity is a common theme. Some educators attempt to revive interest in the subject by working with popular ideas such as fractals; others focus on revisiting 'difficult' topics which are known to be hard to teach and learn.

In the proposals there was a wide variety of task types, but the choice was not always explained. These choices are at the heart of effective teacher education, just as they are at the heart of good teaching. Comparing proofs, diagnosing student errors, comparing solution methods, using controlled variation, modelling, authentic problem solving, comparing representations, creating and adapting tasks, devising practice questions, concept mapping, and matching and sorting are relevant tasks in all mathematical work. What is required is a shared understanding about why, how and when these experiences influence future pedagogy, and why, how and when they are seen as irrelevant for the realities of practice.

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